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Green asset pricing

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Abstract

Climate change is one of the greatest economic challenges of our time. Given the scale of the problem, the question of whether a carbon tax should be introduced is hotly-debated in policy circles. This paper studies the design of a carbon tax when environmental factors, such as air carbon-dioxide emissions (CO2), directly affect agents' marginal utility of consumption. Our first result is that the optimal tax is determined by the shadow price of CO2 emissions. We then use asset-pricing theory to estimate this implicit price in the data and find that the optimal tax is pro-cyclical. It is therefore optimal to use the carbon tax to "cool down" the economy during booms and stimulate it in recessions. The optimal policy not only generates large welfare gains, it also reduces risk premiums and raises the average risk-free real rate. The effect of the tax on asset prices and welfare critically depends on the emission-abatement technology.

Keywords: Climate Change, Compensation Effect, Bond Premium Puzzle, Natural Rate of Interest, Optimal Policy, Welfare.

JEL: Q58, G12, E32.

Non-technical Summary

Climate change is one of the biggest economic challenges of our time. Given the scale of the problem, the question of whether a carbon tax should be introduced is hotly debated in policy circles.

CO2 emissions is a classical example of what economists call "externality". Emissions contribute to climate change, a phenomenon which affects everybody's well-being. The problem is that the adverse effects of emissions are not reflected in market prices. Without a price mechanism, markets fail in the sense that they cannot allocate resources efficiently. This "market failure" in turn leads to excessive CO2 emissions. Government intervention is thus necessary to correct the resulting inefficiency.

This work shows how to design a carbon tax that is optimal from a welfare perspective. We firstly use asset pricing theory to derive the implicit market price of CO2 emissions. We then show that the optimal carbon tax is determined by this implicit price. Next, we use our methodology to compute an estimate of the optimal carbon tax over the business cycle. In our framework, the optimal environmental policy is procyclical. It is therefore optimal to use the carbon tax to "cool down" the economy during periods of booms and to stimulate it in recessions.

Our second main result is that the environmental externality can affect financial markets. In our framework, we find that climate risk reduces the natural rate of interest. This result is relevant for monetary policy because a low natural rate increases the likelihood of hitting the effective lower bound. The reason is that households become more risk averse when firms fail to internalize the damage caused by their emissions. In our simulated economy, low interest rates are thus a consequence of the uncertainty induced by climate change.

A main takeaway is that the effectiveness of the policy critically depends on the ease at which emissions can be abated. The welfare gains from the optimal tax are of a much lower magnitude if the abatement technology is not efficient. Without a well-developed technology, the decline in risk premiums induced by the policy is also much smaller. The success of the policy may therefore critically depend on the timing of implementation. Improving the existing emission abatement technology should probably come first. Once available, an efficient technology would in turn help to mitigate the side effects of the tax, thereby maximizing the welfare gains from the policy.

1 Introduction

Current evidence shows that the mean temperature is 1 degree higher than it was in the pre-industrial era. In recent years, this increase in temperature has accelerated and temperatures are currently estimated to rise by about 0.2 degrees per decade.[1] The link between carbon-dioxide emissions (CO2) and climate change is by now clearly established. CO2 emissions are about 20 times higher than they were at the beginning of the 20th century. Moreover, evidence from Antarctic ice cores shows that CO2 emissions have not only risen rapidly, current levels are also the highest in over 400,000 years.[2]

CO2 emissions are not only a low-frequency phenomenon, they also exhibit large cyclical fluctuations. A decomposition between trend and cyclical components reveals that CO2 emissions are procyclical and more volatile than GDP (e.g. Doda, 2014; Heutel, 2012). Against the background of the ongoing debate over emission taxes, these large cyclical fluctuations raise several important questions. In particular, are these strong cyclical fluctuations desirable from a welfare perspective? And how should the optimal carbon tax vary over the business cycle?

This paper addresses these questions by considering the optimal carbon tax in the presence of an environmental externality. The novelty of our approach is to investigate the link between asset-pricing theory —in particular the stochastic discount factor (SDF)— and climate policies. The SDF is a key building block of modern asset-pricing theory (e.g. Cochrane, 2011). Our main contention is that it also has a critical impact on the design of the optimal carbon tax.

Following Stokey (1998), Acemoglu, Aghion, Bursztyn, and Hemous (2012) and Golosov, Hassler, Krusell, and Tsyvinski (2014), among others, environmental considerations are captured by introducing an externality into the utility function. Apart from a few exceptions (see for instance Michel and Rotillon, 1995), most papers in this literature use a separable specification that implies no direct link between the environment and the marginal utility of consumption. Our innovation is to study a model in which the presence of an environmental externality raises households' willingness to consume goods.

Our approach can be motivated by the effect of climate change on consumption. As documented by Abel, Holloway, Harkey, Meier, Ahl, Limaye, and Patz (2018) and Mansur,

[1] Pachauri, Allen, Barros, Broome, Cramer, Christ, Church, Clarke, Dahe, Dasgupta, et al. (2014).

[2] The Economist (2019). "Briefing Climate Change", Sept. 21st-27th.

Mendelsohn, and Morrison (2008), one perverse effect of climate change is to increase the use of electricity. Higher levels of emissions cause climate change, which in turn increases the need to consume electricity to cool homes. This complementarity between climate change and consumption can be illustrated by the exponential increase in the use of air-conditioning in recent decades.[3] Projections by the International Energy Agency also suggest that this is only the beginning, as the demand for air-conditioning is expected to triple by 2050.[4] This latter result is consistent with the US findings in He, Liang, Qiu, Li, and Xing (2020). Using data for a large sample of consumers, they show that pollution, which is highly correlated with CO2 emissions, increases electricity consumption.

Apart from electricity consumption, there is evidence that emissions also increase other types of expenditure. Deschênes, Greenstone, and Shapiro (2017) show that air pollution increases the consumption of medical products. There is moreover evidence that emissions raise the demand for goods that are used to mitigate the effect of pollution, such as air purifiers (e.g. Ito and Zhang, 2020). Climate change also increases investment in adaptation measures (e.g. Fried, 2019; Gourio and Fries, 2020).

Overall, the evidence therefore suggests the existence of a compensation effect of climate change (e.g. Michel and Rotillon, 1995). As Greenhouse Gas emissions rise, the need to consume electricity as well as other goods to mitigate the effect of climate change becomes more pressing. In other words, the presence of environmental externalities could raise the marginal utility of consumption.

From a finance perspective, this non-separability between consumption and the environmental externality has key implications. Indeed, the SDF —the ratio of future to current marginal utility—is at the core of modern asset-pricing theory. Consequently, if environmental factors modify agents' marginal utility of consumption, they will also affect the pricing of risky and safe assets. This compensation effect of climate change therefore implies a potential role for green factors in asset-pricing models.

We model this compensation effect of climate change via an approach similar to that in the seminal contribution of Campbell and Cochrane (1999). In our case, however, it is the current stock of CO2 emissions rather than past levels of consumption that raises marginal

[3]The Economist (2018). "Air-conditioners do great good, but at a high environmental cost". August 25th.

[4]International Energy Agency (2018). "Air conditioning use emerges as one of the key drivers of global electricity-demand growth". News, May 15th 2018.

utility. Moreover, following Heutel (2012), the stock of emissions is a slow-moving variable whose level depends on the quantity of emissions. As in Campbell and Cochrane (1999), this specification implies that risk aversion increases as the distance between consumption and the externality, or "surplus consumption" in the case of habits, declines. One advantage of this particular specification is that it will allow us to generate realistic fluctuations in the SDF without introducing too many degrees of freedom.

Relative to the endowment economy approach (e.g. Lucas Jr, 1978), another difference is that we analyze the environmental externality in a production economy, following the seminal contribution of Jermann (1998). We then derive the optimal tax by comparing the decentralized equilibrium to the planner's problem, as is usually the case in the environmental literature (e.g. Xepapadeas, 2005) or in Ljungqvist and Uhlig (2000) for the case of a consumption externality.

Following Nordhaus (2008) and Heutel (2012), among others, we introduce an abatement technology that firms can use to reduce their carbon footprint. Even when available, firms do not use this technology if emissions are not taxed. The abatement technology diverts resources from production. Consequently, profit-maximizing firms have no incentive to reduce emissions unless they are forced to do so.

Our first main result is that the optimal tax is determined by the shadow value of CO2 emissions. We show that this implicit price can be expressed as the infinite discounted sum of the marginal disutility caused by emissions. This discounted sum is in turn critically affected by the SDF used by agents to price assets. This result therefore highlights the importance of asset-pricing considerations for the design of an optimal environmental tax.

This link between the optimal tax and the SDF breaks the macro-finance separation (e.g. Cochrane, 2017; Tallarini, 2000). The reason is that the model's ability to reproduce basic asset-pricing moments, such as the bond premium for example, has a crucial impact on the SDF. As the optimal tax is in turn determined by the SDF, the model's financial-market implications affect the design of environmental policies, and hence welfare. In contrast, with a separable preference specification we find that the dichotomy between climate policies and finance is close to perfect.

Imposing a tax on emissions restores the first-best allocation by encouraging firms to use the abatement technology. Abating carbon emissions is costly for firms. From the point of view of the social planner, it is therefore optimal to set the cost of abating emissions that firms face to its implicit market price.

Our second main result is that slow movements in the stock of CO2 can have significant financial-market implications. Of particular relevance to Central Banks is the finding that environmental externalities affect the natural rate of interest. Climate change reduces the natural rate of interest.

The intuition behind this result is that the environmental externality generates time-variation in risk aversion, as in a model with external habits. In other words, when firms fail to internalize the damage caused by their emissions, households become more risk-averse. This rise in risk-aversion raises the risk premium demanded by investors, and induces precautionary saving. This stronger precautionary motive in turn explains the effect on the natural rate of interest.

We next show that introducing an optimal environmental tax reduces risk premia and increases the natural rate of interest. Under our baseline scenario, the tax reduces the premium on a long-term bond by half, and increases the natural rate by around 2 percentage points.

This result can be explained by the effect of the optimal policy on risk aversion. A tax on production reduces output, and hence consumption as well as emissions. The key is that the decline in emissions causes a fall in the externality that exceeds the drop in consumption. The resulting increase in this distance between consumption and the externality in turn reduces risk aversion.

Although consumption declines, the optimal tax generates large welfare gains. Under our benchmark calibration, this result is explained by the large fall in emissions induced by the policy. The magnitude of this gain in turn depends on how firms react to the carbon tax. A profit-maximizing firm increases abatement until the marginal cost of abating emissions equals the marginal benefit. Under the optimal policy, the tax incentivizes firms to use the abatement technology to reduce the burden of the tax. This incentive to reduce emissions therefore lies behind the large welfare gain that we obtain.

The effect on welfare critically depends on the efficiency of the abatement technology available in the economy. If the technology is not sufficiently well-developed, the distortion caused by the tax can be sizeable: if firms cannot circumvent the tax by abating emissions, their only choice is to reduce production. In this case, the tax generates a smaller drop in emissions, which in turn reduces the policy's welfare gains.

The effect of the optimal policy on asset prices also depends crucially on the abatement technology. In this model, this can be explained by the impact of the tax on risk aversion. A less-developed technology reduces the decline in the stock of emission induced by the carbon tax. Consequently, a smaller increase in the distance between consumption and the externality can result if the technology is inefficient. This in turn implies a smaller drop in risk aversion, which causes higher risk premia and lower real interest rates.

Our third main result is that the optimal tax is pro-cyclical. As in Ljungqvist and Uhlig (2000), it is therefore optimal to "cool down" the economy during booms and to stimulate it in recessions. Estimating the model using higher-order perturbation methods allows us to estimate the implicit price of carbon. Our approach can therefore be used to provide an estimate of the optimal carbon tax over the business cycle. As illustrated in Figure 1, it would have been optimal to progressively increase the tax in the run-up to the financial crisis and to reduce it sharply when the financial shock hit.

The intuition here is that the externality produces excessive fluctuations in risk aversion. As in a model with external habits and time-varying risk aversion (e.g. Campbell and Cochrane, 1999), the externality is beyond the agents' control. By internalizing the effect of emissions on utility, the policy allows the planner to find an optimal trajectory for both consumption and the stock of emissions. Controlling both variables at the same time in turn reduces the variations in "surplus consumption" that are unnecessary from a welfare perspective. These lower fluctuations in turn imply more moderate variations in risk aversion.

During recessions, this optimal trajectory involves lowering the carbon tax. A decline stimulates consumption. This effect helps to reduce risk aversion by increasing the distance between consumption and the externality. The key is that, as in the data, the stock of emissions moves very slowly over time. As the impact of the policy on consumption is more immediate, a tax cut generates a rise in consumption that exceeds the increase in the stock of emissions. The optimal policy therefore allows the planner to mitigate the surge in risk aversion that occurs in recessions.

As pointed out by Bansal, Kiku, and Ochoa (2019) and van den Bremer and van der Ploeg (2019), there is evidence that climate-change risk could already be reflected in current equity prices. In Bansal et al. (2019), this link is explored in a model in which climate change is a source of long-run risk (e.g. Bansal and Yaron, 2004). The long-run risk approach relies on Epstein-Zin-Weil preferences (e.g. Epstein and Zin, 1989; Weil, 1989; Weil, 1990).

The results in Bolton and Kacperczyk (2020) also suggest that exposure to carbon emission is already priced-in by investors. They find that the increase in stock returns caused by higher emissions is economically significant. In Van der Ploeg, Hambel, and Kraft (2020), the optimal carbon tax is derived in an endogenous-growth model. They also find that the natural rate of interest is lower under laissez-faire.

Bauer and Rudebusch (2020) show that the decline in the natural interest rate observed over the last decade implies a dramatic increase in the social cost of climate change. Our findings are also related to Gollier (2021) who highlights the role of abatement technologies and their efficiency in shaping carbon pricing. Following Piazzesi, Schneider, and Tuzel (2007), we analyze the asset-pricing implications of a nonseparable utility function. Piazzesi et al. (2007) show that variations in the relative share of housing in agents' consumption baskets is a significant source of risk. In our case, it is the slow movements in the environmental externality that affect marginal utility. A review of the macro-financial implications of climate change is provided by Van der Ploeg (2020).

Our approach also builds on Heutel (2012), which is one of the first papers to consider environmental externalities from a business-cycle perspective. Relative to Heutel (2012), the model is estimated and generates a bond premium of about 1 percent. Reproducing a bond premium of this magnitude is a challenge for standard macroeconomic models (e.g. Rudebusch and Swanson, 2008; Rudebusch and Swanson, 2012). Recent improvements in this literature for instance includes the work of Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2018), which studies feedback effects from long-term bonds to the real economy within a model that matches the level and variability of the term premium.

In our case, environmental factors affect financial markets through the effect of the externality on attitudes towards risk. All else equal, the key is that an increase in the stock of emissions increases risk aversion. While it is difficult to test this hypothesis in the data, recent results in the psychology literature provide some indirect support.

First, in this literature, it is well-established that air pollution tends to increase anxiety. A recent review of the evidence on the link between air pollution and anxiety is provided in Lu (2020). Air pollution is in turn strongly correlated with CO2 emissions. Second, there is evidence that anxiety and risk aversion are tightly linked. For instance, according to Charpentier, Aylward, Roiser, and Robinson (2017), more-anxious individuals exhibit a reduced propensity to take risks. The authors argue that this result is driven by risk aversion, and not loss aversion.

This kind of effect of air pollution on risk aversion is also consistent with the findings in Levy and Yagil (2011) of a negative correlation between air pollution and stock returns. Their interpretation is that air pollution has negative mood effects. As experimental work in Psychology in turn has related bad mood to increased risk aversion, they argue that air pollution could affect stock returns.

2 The model

Consider a business-cycle model characterized by discrete time and an infinite-horizon economy populated by *firms* and *households*, which are infinitely-lived and of measure one. In this setup, production by firms produces an environmental externality via emissions, and these latter affect the household welfare by reducing the utility stemming from the consumption of goods. Firms do not internalize the social cost from their emissions of CO2. As such there is market failure, opening the door to optimal policy intervention.

As the contribution of the paper lies in the role of the environmental externality in shaping investors' risk behavior, we start by presenting the accumulation of emissions in the atmosphere. We then explain how this environmental externality affects households' behavior.

2.1 Balanced growth

Given that one objective of this paper is to estimate the model, we need to take into account that emissions grow at a different rate from output. In the context of our model, this difference in growth rates can be explained by introducing a rate of Green technological progress.

As is standard in the literature, macroeconomic variables are also assumed to grow along the balanced growth path. This is achieved by introducing labor-augmenting technological progress, denoted by Γ_t. The growth rate of labor-augmenting technological progress is γ^Y, where:

$$\frac{\Gamma_{t+1}}{\Gamma_t} = \gamma^Y$$

We denote Green technological progress in the growing economy by Ψ_t. The growth rate of Green progress γ^E is as follows:

$$\frac{\Psi_{t+1}}{\Psi_t} = \gamma^E$$

This trend is necessary to capture the long-term process of the decoupling of output growth from emission growth. As documented by Newell, Jaffe, and Stavins (1999), this trend can be interpreted as an energy-saving technological change that captures the adoption of less energy-intensive technologies in capital goods. An improvement in the technology therefore implies a value for γ^E that is below 1. As in Nordhaus (1991), we assume that this trend is deterministic.

In the following sections, we present the de-trended economy. The detailed derivation of this de-trended economy appears in Appendix C.

2.2 *Firms and emissions*

Following standard integrated assessment models (IAM) (see Nordhaus (1991) and Nordhaus and Yang (1996)), a large part of the accumulation of Carbon Dioxide and other Greenhouse Gases (GHGs) in the atmosphere results from the human activity of economic production. We therefore employ a similar law of motion as in IAM to describe the concentration process of Carbon Dioxide in the atmosphere:

$$\gamma^X x_{t+1} = \eta x_t + e_t, \tag{1}$$

where x_{t+1} is the concentration of gases in the atmosphere, $e_t \geq 0$ the inflow (in kilotons) of Greenhouse Gases at time t, and $0 < \eta < 1$ the linear rate of continuation of CO_2-equivalent emissions that enter the atmosphere on a quarterly basis.[5] Anthropogenic emissions of CO_2 result from both economic production and exogenous technical change:

$$e_t = (1 - \mu_t)\, \varphi_1 y_t^{1-\varphi_2} \varepsilon_t^X. \tag{2}$$

[5]One limitation is that we do not consider emissions from the Rest of the World (ROW). At the same time, US and ROW emissions are strongly correlated at the business-cycle frequency. Moreover, the US accounts for 1/3 of total anthropogenic emissions.

Here, the variable $1 \geq \mu_t \geq 0$ is the fraction of emissions abated by firms, y_t the aggregate production of goods by firms, and variable ε_t^X an AR(1) exogenous shock.

This functional form for emissions allows us to take into account both low- and high-frequency variations in CO_2 emissions. For the high-frequency features of the emissions data, the term $\varphi_1 y_t^{1-\varphi_2}$ denotes the total inflow of pollution resulting from production, prior to abatement. In this expression, φ_1, $\varphi_2 \geq 0$ are two carbon-intensity parameters that respectively pin down the steady-state ratio of emissions-to-output and the elasticity of emissions with respect to output over the last century. While φ_2 is set to 0 in Nordhaus (1991), we follow Heutel (2012) and allow this parameter to be positive to capture potential nonlinearities between output and emissions. Note that for $\varphi_2 < 1$, the emissions function exhibits decreasing returns.

In the de-trended economy, the presence of both Green and labor-augmenting technological progress introduces an adjustment into equation (1), where γ^X is given as follows:

$$\gamma^X = \gamma^E \left(\gamma^Y\right)^{1-\varphi_2}$$

The remaining set of equations for firms is fairly standard, and similar to Jermann (1998). In particular, the representative firm seeks to maximize profit by making a trade-off between the desired levels of capital and labor. Output is produced via a Cobb-Douglas production function:

$$y_t = \varepsilon_t^A k_t^\alpha n_t^{1-\alpha}, \tag{3}$$

where k_t is the capital stock with an intensity parameter $\alpha \in [0,1]$, n_t is labor, and ε_t^A is a total factor productivity shock that evolves as follows: $\log\left(\varepsilon_t^A\right) = \rho_A \log\left(\varepsilon_{t-1}^A\right) + \eta_t^A$, with $\eta_t^A \sim N(0, \sigma_A^2)$. The capital-share parameter is denoted by α. Firms maximize profits:

$$d_t = y_t - w_t n_t - i_t - f\left(\mu_t\right) y_t - e_t \tau_t \tag{4}$$

The real wage is denoted by w_t, $f\left(\mu_t\right)$ is the abatement-cost function, and $\tau_t \geq 0$ a potential tax on GHG emissions introduced by the fiscal authority. Investment is denoted by i_t and the accumulation of physical capital is given by the following law of motion:

$$\gamma^Y k_{t+1} = (1-\delta) k_t + \left(\frac{\chi_1}{1-\epsilon} \left(\varepsilon_t^I \frac{i_t}{k_t} \right)^{1-\epsilon} + \chi_2 \right) k_t \tag{5}$$

where $\delta \in [0,1]$ is the depreciation rate of physical capital and ε_t^I an exogenous shock process, as in Christiano, Motto, and Rostagno (2014). This can be interpreted as an investment shock that captures financial frictions associated with asymmetric information or costly monitoring. As in Jermann (1998), χ_1 and χ_2 are two scale parameters that are calibrated to ensure that adjustment costs do not affect the deterministic steady state of the economy. The elasticity parameter $\epsilon > 0$ measures the intensity of adjustment costs.

The abatement-cost function is taken from Nordhaus (2008), where $f(\mu_t) = \theta_1 \mu_t^{\theta_2}$. In this expression, $\theta_1 \geq 0$ pins down the steady state of the abatement, while $\theta_2 > 0$ is the elasticity of the abatement cost to the fraction of abated GHGs. This function $f(\mu_t)$ relates the fraction of emissions abated to the fraction of output spent on abatement, where the price of abatement is normalized to one.

2.3 Households and the environmental externality

We model the representative household via a utility function where the household chooses consumption expenditures as well as its holdings of long-term government bonds. Following Stokey (1998), Acemoglu et al. (2012) and Golosov et al. (2014), we introduce the environmental externality into the utility function. However, instead of considering an additive specification, we assume that the marginal utility of consumption is affected by the externality.

Given our focus on asset prices, we choose a specification similar to that employed in the seminal contribution of Campbell and Cochrane (1999). As will become clear, adopting this particular specification will dramatically improve the model's ability to generate realistic asset-pricing implications. The difference relative to Campbell and Cochrane (1999) is that it is the disutility caused by pollution rather than past consumption that affects the marginal utility of consumption. As the evolution of x_t is determined by the environmental block of the model (e.g. Nordhaus (1991)), we refer to this preference specification as Campbell and Cochrane/Nordhaus (CCN) preferences.

The utility of the representative agent depends on the distance between consumption and the externality:

$$E_0 \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{(c_t - \phi x_t)^{1-\sigma}}{1-\sigma}, \tag{6}$$

where E_0 is the expectations operator conditioned on information at time 0, $\widetilde{\beta}$ the time discount factor adjusted for growth,[6] and $\sigma > 0$ the curvature parameter. The parameter ϕ represents the sensitivity of utility to a rise in CO_2 concentration in the atmosphere, which is denoted by x_t.[7] This can also be interpreted as the proportion of consumers affected by the damage caused by CO2 emissions. Furthermore, the externality is a predetermined variable that moves slowly over time. This is to account for the possible long-term effects of decisions made in the past, which have possibly irreversible future consequences. This assumption has important implications for optimal choices, which we discuss in the following paragraphs.

First, from a consumer's perspective, consumption and the stock of CO2 emissions can be interpreted as complements. As a result, the marginal utility of consumption increases in CO2 concentration, so that households are more willing to consume when GHG concentration is high. This mechanism, pioneered by Michel and Rotillon (1995), is referred to as the *compensation effect*: households consume as a result of the change in marginal utility following an increase in emissions.

Second, this environmental externality in the utility function has important asset-pricing implications. To illustrate, we define, as in Campbell and Cochrane (1999), the consumption surplus ratio, $s_t = (c_t - \phi x_t) / c_t$. When the surplus falls in cyclical downturns, investors require a higher expected return compared to a standard CRRA utility function with $\phi = 0$. Under these preferences, the coefficient of relative risk aversion is given by $-(u_c''/u_c')c_t = \sigma/s_t$. As such, a higher emissions stock reduces the surplus, which in turn increases risk aversion.

The budget constraint of the representative household is as follows:

$$w_t n_t + b_t + d_t = c_t + p_t^B (b_{t+1} - b_t) + t_t \tag{7}$$

where the left hand-side refers to the household's different sources of income. Total income is firstly comprised of labor income (with inelastic labor supply n_t). Every period, the agent also receives income from holding a long-term government bond, b_t. As the representative

[6]Where $\widetilde{\beta} = \beta\gamma^{1-\sigma}$. See Appendix C for a derivation of the effect of growth on the subjective discount factor.

[7]Note that c_t and x_t do not grow at the same rate in the deterministic steady-state of the model. To obtain a stationary utility function, we assume that, in the growing economy, the preference parameter Θ_t is affected by labor-augmenting and Green technological progress. As we show in Appendix C, this implies the following relationship between ϕ and Θ_t :

$$\Theta_t = \phi \frac{(\Gamma_t)^{\varphi_2}}{\Psi_t}$$

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agent owns firms in the corporate sector, there is last dividend income of d_t.

On the expenditure side, the representative household first spends its income on consumption goods, c_t. The price at which newly-issued government bonds are purchased is p_t^B, and the quantity of new government bonds purchased during the period is $b_{t+1} - b_t$. Finally, we assume that the government levies a lump-sum tax of t_t.

2.4 Government and market clearing

The government finances its expenditures by issuing a bond and collecting taxes. The government budget constraint is as follows:

$$g_t + b_t = p_t^B (b_{t+1} - b_t) + t_t + \tau_t e_t, \tag{8}$$

where public expenditure is denoted by g_t and t_t is a lump-sum tax. The revenue is composed of newly-issued government bonds $b_{t+1} - b_t$ on financial markets to households, while $\tau_t e_t$ denotes the revenues obtained from the implementation of an environmental tax on emissions. In this expression, e_t and τ_t are the level of emissions and the tax, respectively. As in any typical business-cycle model, government spending is exogenously determined and follows an AR(1) process: $g_t = \bar{g}\varepsilon_t^G$, with $\log \varepsilon_t^G = \rho_G \log \varepsilon_{t-1}^G + \eta_t^G$, $\eta_t^G \sim N(0, \sigma_G^2)$, and $\bar{g}$ denoting the steady-state amount of resources that is consumed by the government. This shock accounts for changes in aggregate demand driven by both changes in public spending and the trade balance.

The resource constraint of the economy reads as follows:

$$y_t = c_t + i_t + g_t + f\left(\mu_t\right) y_t. \tag{9}$$

Finally, for the asset-pricing variables, we calculate the risk-free rate and the conditional risk premium respectively as:

$$1 + r_t^F = \left\{ E_t m_{t,t+1} \right\}^{-1}, \tag{10}$$

$$E_t(r_{t+1}^B - r_t^F) = E_t((1 + p_{t+1}^B)/p_t^B - (1 + r_t^F)), \tag{11}$$

where $m_{t,t+1} = \beta^Y \left\{ \lambda_{t+1}/\lambda_t \right\}$ is the stochastic discount factor, and the modified discount factor β^Y is as follows:

$$\beta^Y = \widetilde{\beta}/\gamma^Y$$

3 Welfare theorems with environmental preferences

In this section, we derive the optimal tax by comparing the decentralized equilibrium to the planner's problem.

3.1 The centralized economy

We start by characterizing the first-best allocation and consider the optimal plan that the benevolent social planner would choose so as to maximize welfare. This equilibrium provides the benchmark against which the allocation obtained in the decentralized economy should be compared.

Definition 1 *The optimal policy problem for the social planner is to maximize total welfare in Equation 6 by choosing a sequence of allocations for the quantities* $\{c_t, i_t, y_t, \mu_t, e_t, k_{t+1}, x_{t+1}\}$*, for given initial conditions for the two endogenous state variables* k_0 *and* x_0*, that satisfies equations (1), (2), (3), (5), and (9).*

Define λ_t as the time t marginal utility of consumption, q_t as the shadow value of capital and ϱ_t as the Lagrangian multiplier on the production function (note that both q_t and ϱ_t are expressed in terms of the marginal utility of consumption). The first-order conditions for this problem are as follows:

$$\lambda_t = (c_t - \phi x_t)^{-\sigma}, \tag{12}$$

$$1 = \chi_1 \varepsilon_t^I q_t \left(\varepsilon_t^I \frac{i_{t+1}}{k_{t+1}} \right)^{-\epsilon}, \tag{13}$$

$$q_t = \beta^Y E_t \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \left[(1-\delta_K) + \frac{\chi_1}{1-\epsilon} \left(\varepsilon_{It+1} \frac{i_{t+1}}{k_{t+1}} \right)^{1-\epsilon} + \chi_2 - \chi_1 \left(\varepsilon_{It+1} \frac{i_{t+1}}{k_{t+1}} \right)^{1-\epsilon} \right] + \beta^Y E_t \frac{\lambda_{t+1}}{\lambda_t} \alpha \frac{y_{t+1}}{k_{t+1}} \varrho_{t+1} \tag{14}$$

where:

$$\beta^Y = \widetilde{\beta}/\gamma^Y$$

Letting v_{Et} denote the Lagrange multiplier (expressed in units of marginal utility of consumption) on equation (2), the first-order conditions with respect to the firm's optimal choice of output and abatement are given as follows:

$$\varrho_t + f(\mu_t) + v_{Et}(1-\varphi_2)\, e_t/y_t = 1, \tag{15}$$

$$v_{Et} e_t/(1-\mu_t) = f'(\mu_t)\, y_t. \tag{16}$$

The Lagrange multiplier ϱ_t is usually interpreted as the marginal cost of producing a new good, while v_{Et} is the social planner's value of abatement. Equation (15) thus highlights the key role of emissions in shaping price dynamics: the production of one additional unit of goods increases firm profits but is partially compensated by the marginal cost from abating emissions. The planner also takes into account the marginal cost from emitting GHGs in the atmosphere. Notice that if abatement effort is zero, the marginal cost of production is one, as in the standard real business-cycle model. The second equation (16) is a standard cost-minimizing condition on abatement: abating CO2 emissions is optimal when the resulting marginal gain (the left-hand side of equation 16) is equal to its marginal cost (the right-hand side of the same equation).

Two remaining first-order conditions on each of the environmental variables, namely x_t and e_t, are necessary to characterize the decision rules of the social planner:

$$v_{Xt} = \beta^X E_t \frac{\lambda_{t+1}}{\lambda_t}(\phi + \eta v_{Xt+1}) \tag{17}$$

$$v_{Et} = v_{Xt}. \tag{18}$$

where:

$$\beta^X = \widetilde{\beta}/\gamma^X$$

Recall that v_{Et} is the Lagrange multiplier on emissions in equation (2), while v_{Xt} is the Lagrange multiplier on the law of motion of GHGs in equation (1). The variable v_{Xt} can be interpreted as the implicit price of carbon. Equation (17) shows that this implicit price can be considered via an asset-pricing formula. The first term- $(\beta^X E_t \frac{\lambda_{t+1}}{\lambda_t}\phi)$ is the discounted utility loss incurred by society from a marginal increase in the stock of emissions in the atmosphere. The second term $(\eta\{E_t \frac{\lambda_{t+1}}{\lambda_t} v_{t+1}^X\})$ is the continuation value of the discounted utility loss caused by emissions, which remain in the atmosphere with probability η. The second equation is the internal cost of GHG emissions for firms, where v_{Et} is the marginal cost for a firm of emitting one kiloton of carbon. In the first-best allocation, this cost must be exactly equal to the price of carbon emissions v_{Xt}.

Definition 2 *The inefficiency wedge induced by the environmental externality is defined as the gap between the price of carbon emissions and this marginal cost:*

$$\varpi_t = v_{Xt} - v_{Et}. \tag{19}$$

When the social cost of carbon is perfectly internalized by society, optimal abatement in (18) is such that the marginal cost of emissions equals their price. In this case, it is optimal for firms and society to spend a fraction of resources to reduce CO2 emissions by using the abatement technology $f(\mu_t)$.

Proposition 1 *In a centralized equilibrium, the social cost of carbon is perfectly internalized by the planner. The marginal cost of emissions is therefore equal to the price of carbon emissions. This implies (from the previous definition) a first-best allocation with an inefficiency wedge* $\varpi_t = 0$*.*

The resulting equilibrium is optimal, as the social cost of the externality is perfectly internalized by society. As a consequence, the inefficiency wedge from carbon emissions is zero. In the following section, we show that this optimum is not reached in a *laissez-faire* equilibrium with profit-maximizing firms.

3.2 The competitive equilibrium

We now describe the competitive equilibrium resulting from economic decisions taken by households and firms separately, with no centralization. This decentralized economy is also referred to as the competitive or *laissez-faire* equilibrium, where social preferences for carbon are different across firms and households. We propose the following definition to characterize this economy.

Definition 3 *The laissez-faire equilibrium is defined as a competitive equilibrium in which the environmental tax on carbon emissions τ_t is set to 0. Households maximize utility in Equation 6 under constraints (7) and (5). Firms maximize profits (4) under constraints (2) and (3).*

Relative to the efficient equilibrium, the difference here is that firms maximize profits and no longer consider the stock of CO2 emissions as a control variable. This implies that firms and households exhibit different preferences regarding carbon emissions. As a result, the price of carbon for firms differs from that obtained in the centralized economy. Since emissions are costly to abate, and given that firms do not internalize the effect of their emissions on consumers, the cost of carbon emissions for firms is zero. In contrast, the price of carbon for households, which we denote v_{Xt}, is given as follows:

$$v_{Xt} = \beta^X E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\phi + \eta v_{Xt+1}\right) \tag{20}$$

We here have a market failure, as the social value of carbon differs between the emitters of carbon and the agents who experience the social loss.

As emissions are not taxed, the shadow cost for a firm to emit CO2 in the atmosphere is zero:[8]

$$v_{Et} = 0. \tag{21}$$

In this setup, firms simply cost-minimize by optimally choosing zero abatement spending: with a cost of releasing CO2 of zero, firms have no incentive to allocate resources to use the abatement technology $f(\mu_t)$ to reduce emissions. The socially-optimal level of abatement is not implemented, as the equilibrium abatement share is zero in the *laissez-faire* equilibrium:

$$\mu_t = 0. \tag{22}$$

Consequently, the marginal cost of production ϱ_t is similar to that obtained in any typical real business-cycle model. In terms of the notation introduced in definition 3, this produces an environmental inefficiency wedge that differs from zero:

$$\varpi_t = v_{Xt} - v_{Et} = v_{Xt}. \tag{23}$$

[8]The optimality conditions corresponding to the *laissez-faire* equilibrium are derived in Appendix D.

CO2 emissions therefore create a market failure via an environmental externality. As a result, the first welfare theorem breaks down as the competitive equilibrium does not coincide with the social planner's outcome. The externality, measured by the inefficiency wedge ϖ_t, distorts the equilibrium and gives rise to a deadweight loss proportional to υ_{Xt}. Note that the first welfare theorem applies only if the environmental policy has no effect on preferences, which is the case only if $\phi = 0$.

3.3 Environmental policy

In the presence of the environmental externality reflected in $\varpi_t > 0$, the social value of carbon differs across agents. This market failure opens the door for government policy to address this externality and render the *laissez-faire* allocation the same as that of the social planner. In particular, the government can introduce a tax, τ_t, on GHG emissions to be paid by firms. This policy tool has two interpretations. It first can be considered as a tax on carbon emissions, in the same spirit as a standard Pigouvian tax that aims to force firms to internalize the social cost of carbon emissions on household utility, thereby correcting the market failure (i.e. the negative externality) by setting the tax equal to the price of carbon emissions.

An alternative interpretation is that the government creates a market for carbon emissions (i.e. a carbon-permits market). Here the government regulates the quantity of emissions. The optimal value for this instrument can be directly computed from a Ramsey optimal problem. Comparing the social planner's solution to the competitive equilibrium, we make the following proposition:

Proposition 2 *The first-best allocation can be attained by using the instrument τ_t in order to close the inefficiency gap (i.e. $\varpi_t = 0$). This condition is achieved by setting the carbon tax such that:*

$$\tau_t = \upsilon_{Xt}.$$

As shown in Appendix D, setting the environmental tax to υ_{Xt} ensures that the first-order conditions under the competitive and centralized equilibria coincide. This result is fairly intuitive. In the absence of an environmental policy, abatement reduces profits, and firms will not be willing to bear this cost unless an enforcement mechanism is implemented. The government can impose a price on carbon emissions by choosing the optimal tax (either quantity- or price-based, as discussed in Weitzman (1974)) to produce the desired level of abatement. This environmental policy forces firms to internalize the effect of emissions, which in turn leads to a better integration of economic and environmental policies.

Furthermore, as argued in both the public economics and environmental literatures (Goulder (1995)), either a tax or a permit policy would generate revenue that could be used as a "double dividend" to not only correct the externality but also reduce the number of distortions due to the taxation of other inputs, such as labor and capital. Moreover, an equivalence between the tax and permit policies holds when the regulator has symmetric information about all state variables for any outcome under the tax policy and a cap-and-trade scheme (Heutel (2012)).

4 Estimation

In this section, we estimate the structural parameters of the model using Bayesian methods. For a presentation of the method, we refer to the canonical papers of An and Schorfheide (2007) and Smets and Wouters (2007). As the U.S. has not implemented any environmental policy, we propose to estimate the *laissez-faire model*. The following sub-sections discuss the non-linear method employed for the estimation, the data transformation and calibration, the priors and the posteriors.

4.1 Solution method

Since we want to accurately measure higher-order effects of environmental preferences (e.g. precautionary saving, utility curvature), we consider a second-order approximation to the decision rules of our model. Taking higher-order approximated models to data remains a challenge as the nonlinear filters that are required to form the likelihood function are computationally expensive. An inversion filter has recently emerged as a computationally-cheap alternative to apply nonlinear models to data (e.g. Guerrieri and Iacoviello 2017, Atkinson, Richter, and Throckmorton 2020). Initially pioneered by Fair and Taylor (1987), this filter extracts the sequence of innovations recursively by inverting the observation equation for a given set of initial conditions. Unlike other filters (e.g. Kalman or particle),[9] the inversion

[9] For a presentation of alternative filters to calculate the likelihood function, see Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016).

filter relies on an analytic characterization of the likelihood function. Kollmann (2017) provided the first application of the inversion filter to second- and third-order approximations to the decision rules in a rational-expectations model.[10] To allow the recursion, this filter imposes that the number of fundamental shocks must be equal to the number of observable variables. Note that, for linearized models, this restriction is standard following Smets and Wouters (2007). For the relative gains of the inversion filter with respect to a particle filter, we refer to Cuba-Borda, Guerrieri, Iacoviello, and Zhong (2019) and Atkinson et al. (2020).

The model is estimated using four observable macroeconomic time-series, which are jointly replicated by the model through the joint realization of four corresponding innovations. Note that we use the pruning state-space to obtain the matrices of the policy rule using the Dynare package of Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011). From this state-space representation, we reverse the observation equations to obtain the sequence of shocks. Unlike Kollmann (2017) who limits the analysis to a frequentist approach, we augment the likelihood function with prior information in the same spirit as Smets and Wouters (2007). This method requires a sampler, here Metropolis-Hastings, to draw the parametric uncertainty.

4.2 Data

The model is estimated with Bayesian methods on U.S. Quarterly data over the sample time period 1973Q1 to 2018Q4, which are all taken from FRED and the U.S. Energy Information Administration.

Concerning the transformation of series, the aim is to map non-stationary data to a stationary model (namely, GDP, consumption, investment and CO2 emissions). Following Smets and Wouters (2007), data exhibiting a trend or unit root are rendered stationary in two steps. We first divide the sample by the working-age population. Second, data are taken in logs and we apply a first-difference filter to obtain growth rates. Real variables are deflated by the GDP deflator price index. The measurement equations mapping our model

[10] Kollmann (2017) posits a modified higher-order decision rule in which powers of exogenous innovations are neglected to obtain a straightforward observation equation inversion. In this paper, we include these terms of the decision rule.

to the data are given by:

$$\begin{bmatrix} \text{Real Per Capita Output Growth} \\ \text{Real Per Capita Consumption Growth} \\ \text{Real Per Capita Investment Growth} \\ \text{Per Capita } CO_2 \text{ Emissions Growth} \end{bmatrix} = \begin{bmatrix} \log \gamma_A + \Delta \log (\tilde{y}_t) \\ \log \gamma_A + \Delta \log (\tilde{c}_t) \\ \log \gamma_A + \Delta \log (\tilde{\imath}_t) \\ \log \gamma_A^{1-\varphi_2} \gamma_E + \Delta \log (\tilde{e}_t) \end{bmatrix}, \tag{24}$$

where a variable with a tilda, $\tilde{x}_t$, denotes the de-trended version of a level variable, x_t.

4.3 Calibration and prior distributions

The calibrated parameters are reported in Table 6. The calibration of the parameters related to business-cycle theory is standard: the depreciation rate of physical capital is set at 2.5 percent in quarterly terms, the Government spending to GDP ratio to 20 percent, and the share of hours worked per day to 20 percent. The environmental component parameters of the models, when not estimated, are set in a similar fashion as Nordhaus (2008) and Heutel (2012). We set the parameter φ_1 to match an average steady-state of pollution in laissez-faire equilibrium, which corresponds to the 2005 value of atmospheric carbon mass of 800 gigatons. The continuation rate of carbon in the atmosphere, denoted η, is set to match a roughly 139 years half time of atmospheric carbon dioxide, as in Nordhaus (1991).[11] Finally, for the abatement-cost function, we set $\theta_1 = 0.05607$ and $\theta_2 = 2.8$ as in Nordhaus (2008) and Heutel (2012).

For the remaining set of parameters and shocks, we employ Bayesian methods. Table 7 summarizes the prior — as well as the posterior — distributions of the structural parameters for the U.S. economy. Let us first discuss the prior for structural disturbances. The prior information on the persistence of the Markov processes and the standard deviation of innovations are taken from Guerrieri and Iacoviello (2017). In particular, the persistence of shocks follows a beta distribution with a mean of 0.5 and a standard deviation of 0.2, while

[11] Let us assume that each unit of CO2 is subject to an idiosyncratic shock, denoted ω, and that the carbon is reused or sequestered in a carbon sink. This random variable is drawn from a binomial distribution, $\omega \sim B(n,p)$ with n the number of trials and p the probability of success $p = 1 - \tilde{\eta}$. We thus determine the number of trials, n, that are necessary on average for one unit of carbon to be sequestrated. Recall that $E(\omega) = n.p$, by imposing $E(\omega) = 1$ we calculate that the average number of trials necessary for carbon sequestration is $n = 1/(1-\tilde{\eta})$. On an annual basis, the latter becomes $n = 0.25/(1-\tilde{\eta})$. Recall that in the balanced growth path the effective continuation rate of carbon is $\tilde{\eta} = \eta \gamma_A \gamma_E^{1-\varphi_2}$. Then imposing an average half time of carbon of 139, we deduce the value of η as $\tilde{\eta} = (1 - 0.25/139)(\gamma_A \gamma_E^{1-\varphi_2})^{-1}$.

for the standard deviation of shocks we choose an inverse gamma distribution with mean 0.01 and standard deviation of 1.

For the parameters which have key asset-pricing implications, we translate some bound restrictions from the matching moments exercise of Jermann (1998) into prior distributions. In particular, the elasticity of Tobin's Q to the investment-capital ratio is assumed to follow a Gamma distribution with prior mean of 4 and standard deviation of 1. The latter implies a support for ϵ close to the bound $\epsilon \in [0.16; +\infty]$ of Jermann (1998). In addition, we set the capital intensity α to follow a Beta distribution with mean of 0.25 and standard deviation 0.02 in order to be close to the value estimated by Jermann (1998). Note that we set a tight prior on this parameter in order to match the tight interval range of α that replicates the U.S. investment-to-output ratio. Jermann (1998) calibrates the risk aversion coefficient to 5 to be consistent with asset-pricing models. However, a high value for σ typically generates strong consumption-smoothing behavior in the Euler equation that is at odds with the data. Environmental economics typically favors values close to 2, while likelihood-estimated models usually find values below 2 (e.g. Smets and Wouters (2007)). To reconcile these three literatures, we propose to estimate this key parameter agnostically by imposing a rather diffuse information through a Gamma distribution with a prior mean of 2 and standard deviation of 0.35. This prior allows the parameter to be either high (i.e. close to 5), as in asset-pricing models, or lower (i.e. close to 2), as in the environmental models in Stern (2008) and Weitzman (2007), or low (i.e. equal to one), as in estimated business-cycle models. Unlike Jermann (1998), we cannot directly estimate $\beta\gamma_A^{-\sigma}$, because of weak identification when using full-information methods. We thus follow Smets and Wouters (2007) and estimate instead the term $(1/\beta - 1)100$: this allows to easily impose prior information based on a Gamma distribution with a mean of 0.5 and standard deviation 0.25. The resulting prior allows the discount factor to roughly lie between 0.99 and 0.9980.[12]

Regarding the slopes of growth, we discuss first the productivity one (denoted $(\gamma_A - 1) \times 100$) that follows a Gamma distribution with a prior mean of 0.5 and a standard deviation of 0.04 in order to match the average 0.40 percent quarterly growth rate. For the (de)coupling rate (denoted $(\gamma_E - 1) \times 100$), we let the data be fully informative about the slope through

[12]Note in addition that our prior mean for $(1/\beta - 1)100$ is much higher than that in Smets and Wouters (2007) as our model is non-linear, and thus features the precautionary saving effect that drives down the real rate. With the prior information of Smets and Wouters (2007), we would obtain a real rate below zero; we thus re-adjust the prior information to render our non-linear model consistent with US real rate data.

a normal distribution with prior mean 0 and standard deviation 0.25. Finally, the last remaining parameter is the utility loss from cumulative CO2 emissions, ϕ. As in Campbell and Cochrane (1999), and given that we have several exogenous shocks, this parameter has to be restricted to ensure that surplus consumption always remains positive. This restriction ensures the non-negativity of the Lagrangian multiplier on the budget constraint (otherwise the budget constraint would not bind). We thus express this parameter in terms of steady-state consumption, $\phi\bar{c}/\bar{x}$, and impose an uninformative prior with an uniform distribution with mean 0.5 and standard deviation 0.285. This prior induces a bound restriction such that $\phi\bar{c}/\bar{x} \in [0; 1]$, this is rather conservative as, unlike Beta distributions, it does not favor any particular value within this interval.[13]

4.4 Posterior distributions

In addition to prior distributions, Table 7 reports the means and the 5th and 95th percentiles of the posterior distributions drawn from four parallel MCMC chains of 50,000 iterations each. The sampler employed to draw the posterior distributions is the Metropolis-Hasting algorithm with a jump scale factor so as to match an average acceptance rate close to 25-30 percent for each chain.

The results of the posterior distributions for each estimated parameter are listed in Table 7 and Figure 2. It is clear from Figure 2 that the data were informative, as the shape of the posterior distributions is different from the priors. Our estimates of the structural parameters that are common with Smets and Wouters (2007) are mostly in line with those they find. The persistence of productivity and spending shocks are, for instance, very similar to theirs. The risk-aversion coefficient σ has a posterior mean of 4.2, which is lower than the value in Jermann (1998). It is however higher than the values reported in environmental macroeconomic and estimated DSGE models: for example, Smets and Wouters (2007) find a value of 1.38 for this parameter. Another key parameter that determines the consumption surplus is $\phi\bar{c}/\bar{x}$. We find a value of 0.67 which is very close to that estimated by Smets and Wouters (2007) in the case of external consumption habits (0.71). The corresponding value of ϕ, given the steady state ratio $\bar{c}/\bar{x}$, is 0.0004. Regarding the growth rate of productivity, our estimated value, 0.34, is lower than that in Smets and Wouters (2007), but this is

[13] Note that with the bounds $\hat{\phi} = \phi\bar{c}/\bar{x} \in [0; 1)$, the MRS$=\bar{c} - \phi\bar{x} = \bar{c} - \hat{\phi}\bar{c}$, as in any standard model featuring external consumption habits.

unsurprising as economic growth is lower in our sample given that we exclude the 1960s and include the last decade. Regarding the last estimated parameter common with Smets and Wouters (2007), the data suggest a value for capital intensity α close to 0.41, which is higher than the estimated values of Jermann (1998) and Smets and Wouters (2007). This is important, as estimated DSGE models typically predict very low values for α that are at odds with data on both the capital structure of firms and the investment-to-output ratio. Finally for the discount rate, denoted $100\left(\beta^{-1}-1\right)$, we find a posterior mean of 0.13 that generates a discount factor of 0.9987.

The last remaining parameters are not common with Smets and Wouters (2007). For the elasticity of Tobin's Q to the investment capital ratio ϵ, we find a posterior mean of 1.44 that is higher than that in Jermann (1998). The value of the elasticity of emissions to output, φ_2, is 0.36, which is remarkably close to that estimated by Heutel (2012). Finally, for the decoupling rate we find that energy-saving technological change has caused reductions in CO2 of about 2% annually.

	Mean		Stand. Dev		Corr. w/ output	
	Data [5%;95%]	Model	Data [5%;95%]	Model	Data [5%;95%]	Model
$100 \times \Delta \log(y_t)$	[0.28;0.50]	0.34	[0.69;0.85]	0.81	[1.00;1.00]	1.00
$100 \times \Delta \log(c_t)$	[0.36;0.55]	0.34	[0.60;0.74]	0.90	[0.54;0.76]	0.58
$100 \times \Delta \log(i_t)$	[0.07;0.68]	0.34	[1.91;2.34]	2.58	[0.61;0.80]	0.72
$100 \times \Delta \log(e_t)$	[-0.53;0.07]	-0.24	[1.88;2.31]	2.12	[-0.01;0.35]	0.24

Table 1: Data moments vs. model moments (with parameters taken at their posterior means)

To assess the relevance of the estimated model, as in Jermann (1998), we compare the observable moments taken at a 90 percent interval versus the asymptotic moments generated by the model using a second-order approximation to the policy function. Table 1 reports the results. We find that our model does a reasonably good job at replicating some salient features of the data, as most of the moments simulated by the estimated model fall within the 95 percent confidence interval of the data.

The advantage of using Bayesian estimation is that the model can replicate the historical path of the observable variables that we introduce. Once the shock process parameters have been estimated, it is possible to simulate the model by drawing shocks from the estimated distribution. As illustrated in Table 1, however, this procedure does not ensure that the unconditional standard deviations observed in the data are matched perfectly.

	Standard Cons. habits	Pollution externality
Utility function $u(c_t - \mathcal{C}_t)$	$\mathcal{C}_t = \phi c_{t-1}$	$\mathcal{C}_t = \phi x_t$
Surplus parameter ϕ	0.21	0.67
Prior probability	0.50	0.50
Log marginal data density	2004.77	2045.99
Bayes ratio	1.0000000000	8.02e17
Posterior model probability	0.0000000000	1.0000000000

Table 2: The comparison of prior and posterior model probabilities in the internal consumption habits and the environmental preferences models (with parameters taken at their posterior mode).

Letting $u(c_t - \mathcal{C}_t)$ denote the utility function, with $\mathcal{C}_t$ the reference variable to calculate the surplus consumption ratio, a natural question at this stage is how relevant is our specification of environmental preferences with respect to a standard consumption habits model *à la* Jermann (1998). Using an uninformative prior distribution over models (i.e. 50% prior probability for each model), Table 2 shows both the posterior odds ratios and model probabilities taking the consumption habits model $\mathcal{M}\,(\mathcal{C}_t = \phi c_{t-1})$ as the benchmark model. We examine the hypothesis H_0: $\mathcal{C}_t = \phi c_{t-1}$ against the hypothesis H_1: $\mathcal{C}_t = \phi x_t$. The posterior odds of the null hypothesis of surplus based on lagged consumption is 8e17: 1, which leads us to strongly reject the null. The surplus consumption ratio is therefore more relevant when it is based on the stock of emissions rather than past consumption. This result should however be qualified, as prior distributions were selected here to estimate our model and do not necessarily fit the benchmark model of H_0. This can diminish the empirical performance of the benchmark. The goal of this exercise is not to show that one model outperforms another, but to highlight that our model is least as consistent with the data as the standard habits-type model.

5 Results

Our main simulation results appear in Table 3 below. The top panel of this table shows the average level of consumption and the stock of CO2 emissions, which are denoted by $E(c_t)$ and $E(x_t)$, respectively. The agent's lifetime utility, $E(\mathcal{W}_t)$, is our measure of welfare. The average tax chosen by the social planner is $E(\tau_t)$.

The asset-pricing implications appear in the middle panel, where $400E(r_t^F)$,

$400E\left(r_{t+1}^B - r_t^F\right)$ and $std(\hat{\lambda}_t)$ are the mean real risk-free rate, the mean bond premium, expressed in annualized percent, and the standard deviation of marginal utility respectively. The average coefficient of relative risk aversion is $E(RRA_t)$ and $std(\widehat{rra}_t)$ is a measure of its standard deviation (expressed in log-deviations from the steady state).

The bottom panel of Table 3 first lists the share of emissions that firms choose to abate, $E(\mu_t)$. The average cost of abatement is $E(f(\mu_t))$, and $E(\tau_t e_t/y_t)$ is the average cost of the tax borne by firms as a share of GDP.

The first column shows these model implications in the decentralized *laissez-faire* equilibrium with a tax set to zero. Columns (2) to (4) show what happens once the optimal tax is introduced. The optimal-policy results are listed for three different values of the parameter θ_1. This latter measures the efficiency of the abatement technology, with higher θ_1 corresponding to a less-efficient technology. As $\theta_1 = 0.05607$ is the value used in the literature (e.g. Nordhaus 2008; Heutel 2012), the results in column (2) correspond to our baseline scenario.

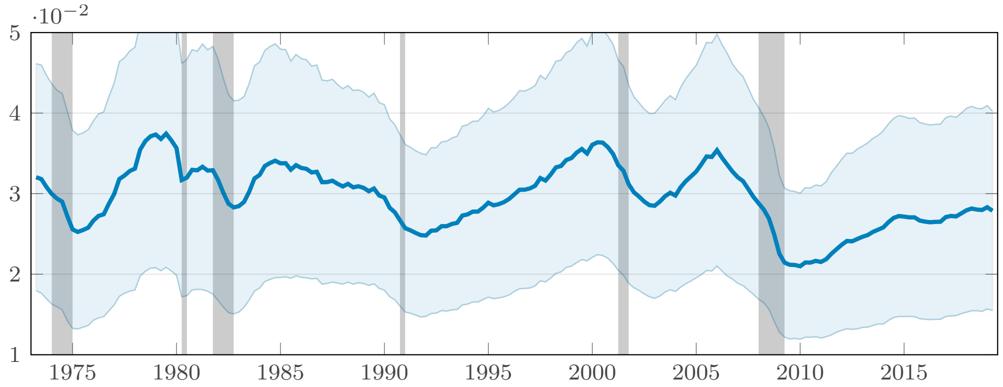
5.1 The size and the cyclicality of the optimal tax

The first main takeaway from Table 3 is that a small average carbon tax is sufficient to restore the first-best allocation. In our benchmark scenario, which corresponds to $\theta_1 = 0.05607$, the total tax bill is on average around two percent of GDP ($E(\frac{\tau_t e_t}{y_t}) = 0.02$).

As can be seen by comparing the total tax bill across columns 2 to 4, in the worst-case scenario, corresponding to a value for θ_1 implying a very-inefficient abatement technology, the total tax bill rises to 5.7 percent of GDP. In this adverse scenario, firms only manage to abate about 6 percent of all emissions, $E(\mu_t) = 0.0592$, once the tax is introduced.

One advantage of our method is that it can be used to construct counterfactual scenarios. In particular, we can answer the following question: What would the optimal tax τ_t have been in the United States from 1973 to 2018, had this optimal policy been implemented? Figure 1 provides the answer. The optimal tax is time-varying, and rises in booms and falls during recessions. The optimal tax is thus strongly pro-cyclical, as illustrated by Figure 3, so that the tax bill $\tau_t e_t/y_t$ falls during major recessions, like the global financial crisis.

The optimal tax is pro-cyclical because the externality induces excessive fluctuations in risk aversion. As in a model with external habits and time-varying risk aversion (e.g. Campbell and Cochrane 1999), agents take the externality as given. As the optimal tax reproduces the first-best allocation, it eliminates this inefficiency by making firms internalize the effect of their production on consumers. Our analysis therefore provides a novel interpretation of the result in Ljungqvist and Uhlig (2000) for the case of habits. As shown in Table 2, one motivation for our approach is that our specification is strongly supported by the data, especially relative to habits.



Notes: The simulated path is expressed in levels. The blue shaded area is the parametric uncertainty at 95% confidence level, drawn from 1,000 Metropolis-Hastings random iterations. The blue line represents the mean of these 1,000 simulated paths. The gray shaded areas are NBER-dated recessions in the US.

Figure 1: Historical variations in the environmental tax

It is important to note that the fluctuations in risk aversion are essentially driven by consumption, not by the externality. In line with the evidence, we assume that the stock of CO2 depreciates very slowly over time. Whereas the flow of emissions can be volatile, the stock of emissions, and hence the externality, moves only very slowly over the business cycle.

5.2 The risk premium and the risk-free rate in the laissez-faire equilibrium

As can be seen in column (1), the model generates an average bond premium, i.e. $400E\left(r_{t+1}^{B}-r_{t}^{F}\right)$, of about 1.3 percent. Although small, generating a bond premium of this magnitude remains a challenge for a large class of General-Equilibrium models with production. In our case, this relative success is due to our preference specification, which generates time-variation in risk aversion, as in Campbell and Cochrane (1999).

As in Jermann (1998), the positive bond premium that we obtain is due to interest-rate risk. The price of long-term bonds is determined by the term structure of interest rates. The key is that in this model short- and long-term interest rates are counter-cyclical. With interest rates rising during recessions, bond holders can expect capital losses to occur precisely during periods of low consumption and high marginal utility. Long-term bonds are therefore not good hedges against consumption risk. The positive bond premium is thus a compensation for holding an asset whose price declines during periods of low consumption.

In this model, the mean risk-free rate $400E(r_t^F)$ is critically affected by uncertainty. As in Jermann (1998), a greater variance in marginal utility reduces the unconditional mean risk-free rate. The intuition is that a higher volatility of marginal utility implies more uncertainty about future valuations, and greater uncertainty in turn increases agents' willingness to build precautionary buffers. This effect therefore captures the impact of this precautionary motive on equilibrium interest rates.

5.3 Asset prices under the optimal policy

Relative to the *laissez-faire* equilibrium, the optimal tax has a sizeable effect on the mean risk-free rate. In the baseline scenario, under optimal taxation, our model predicts a rise in the average risk-free rate of around 2 percent. This effect on the risk-free rate can be better understood by comparing the volatility of marginal utility $std(\hat{\lambda}_t)$ in the two cases. One main effect of the tax is to reduce the volatility of marginal utility. Fluctuations in marginal utility provide a measure of uncertainty about future valuations. The lower volatility therefore reflects that agents face less uncertainty after the introduction of the tax. The higher mean risk-free rate can therefore be interpreted as reducing agents' precautionary saving motives.

The second effect of the tax is to reduce the risk premium. This can be explained by the effect of the tax on risk aversion. The carbon tax reduces both consumption and the stock of emissions, with the reduction in the latter being larger. The distance between consumption and the externality therefore rises. In this model, a larger gap between consumption and the externality in turn reduces risk aversion.

In contrast to an endowment economy, in our production economy lower risk aversion affects the dynamics of consumption as it implies a higher elasticity of intertemporal substitution (EIS). In other words, agents' consumption-smoothing motives are reduced under the optimal policy. This willingness to tolerate larger fluctuations in consumption has in turn asset-pricing implications. As agents are less reluctant to reduce consumption during recessions, there is less need to insure against such outcomes. Consequently, the premium required to compensate investors for holding an asset the price of which falls in recessions is also lower.

5.4 Welfare analysis

To assess the welfare implications of the optimal policy, Table 3 also shows agents' lifetime utility $E(\mathcal{W}_t)$, where:

$$E(\mathcal{W}_t) = E_0 \left\{ \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{(c_t - \phi x_t)^{1-\sigma}}{1-\sigma} \right\}$$

As can be seen by comparing the value of $E(\mathcal{W}_t)$ across columns (1) and (2), the policy generates a sizeable rise in welfare. This welfare gain illustrates that the fall in the stock of emissions $E(x_t)$ more than compensates for the lower average consumption the tax produces. This result highlights the importance of the elasticity of emissions to a change in the tax. As this elasticity depends on firms' willingness to reduce emissions, we now discuss the role of the abatement technology.

5.5 The role of the abatement technology

The purpose of columns (3) and (4) is to illustrate that the effect of the optimal tax critically depends on the efficiency of the abatement technology. In the laissez-faire equilibrium, the externality not being internalized leads firms to spend nothing on abatement. By forcing firms to internalize the externality, the tax incentivizes firms to use the abatement technology to reduce the burden of the tax.

In our preferred scenario, about 55 percent of emissions are abated once the optimal tax is introduced. As shown in the bottom panel of Table 3, when θ_1 is above 0.056, less-efficient technology reduces the share of emissions abated $E(\mu_t)$. Note that as abatement-technology efficiency declines, the planner also chooses to allocate a larger fraction of resources to consumption. This reflects that this model embeds a trade-off between consumption and the abatement technology. The marginal cost of renouncing a unit of consumption should equal the marginal benefit from abating one unit of emissions. Consequently, the planner finds it optimal to allocate more resources to consumption as abatement-technology efficiency falls.

As can be seen by comparing $E(\mathcal{W}_t)$ across columns (2) to (4), the size of the welfare gain depends critically on the abatement technology. This illustrates that the distortion caused by the tax can be sizeable if the technology is not sufficiently well-developed. If emissions are costly to abate, the policy has a stronger negative impact on production, as it is more

	Laissez-faire	Optimal policy		
	Estimation (1972-2019)	$\theta_1 = 0.05607$	$\theta_1 = 0.28844$	$\theta_1 = 3.4996$
	(1)	(2)	(3)	(4)
		Business-cycle variables		
$E(c_t)$	0.5502	0.5206	0.5310	0.5409
$E(x_t)$	848.9287	380.1978	632.2172	777.2627
$E(\mathcal{W}_t)$	-206778.4449	-10649.9577	-43607.6821	-124258.5811
$E(\tau_t)$	0.0000	0.0353	0.0390	0.0433
$std(\tau_t)$	0.0000	0.0063	0.0083	0.0101
		Asset-pricing implications		
$400E\left(r_t^F\right)$	3.5870	5.4100	4.7417	4.0028
$400E\left(r_{t+1}^B - r_t^F\right)$	1.1542	0.6432	0.9176	1.1432
$std(\hat{\lambda}_t)$	2.4445	1.2753	1.7525	2.1893
$E(RRA_t)$	32.1922	12.9862	19.6120	27.1408
$std(\widehat{rra}_t)$	0.5837	0.3045	0.4185	0.5228
		Abatement technology		
$E(\mu_t)$	0.0000	0.5269	0.2234	0.0592
$E(f(\mu_t))$	0.0000	0.0094	0.0044	0.0013
$E(\frac{\tau_t e_t}{y_t})$	0.0000	0.0233	0.0423	0.0566

Notes: The first column is the estimated model under the laissez-faire equilibrium, with no abatement and no environmental tax. Column (2) is the equilibrium under an environmental tax with θ_1 set as in the literature. Columns (3) and (4) are equilibria under alternative values of θ_1 that match an abatement share $\bar{\mu}$ of 20% and 5%. Note that $E(\mu_t) \neq \bar{\mu}$ in columns (3) and (4), due to the contribution of future shocks to the asymptotic mean of these variables.

Table 3: In column (1), the model simulations correspond to the laissez-faire equilibrium. The simulations under the optimal environmental policy are shown in columns (2) to (4). Columns (2) to (4) correspond to different abatement costs, ranging from low to high.

difficult for firms to circumvent the tax. In this case, the tax generates a smaller drop in emissions, which in turn reduces the policy's welfare gains.

Comparing the effect of the optimal tax on $400E\left(r_t^F\right)$ and $400E\left(r_{t+1}^B - r_t^F\right)$, the effect on asset prices also depends crucially on θ_1. Relative to the first-best scenario, the effect of the tax on the risk premium is more muted when the abatement technology is less efficient.

This illustrates that part of the reduction in uncertainty is due to the additional margin provided by the abatement technology. The effect of θ_1 is therefore akin to the adjustment-cost parameter in Jermann (1998). The more efficient is the abatement technology, the easier it is for agents to insure against unexpected shocks. This greater flexibility makes the economy less risky from a consumption-smoothing perspective, which reduces the risk premium and increases the risk-free rate.

5.6 The coefficient of relative risk aversion

Table 3 also lists the average level of risk aversion, where risk aversion is defined as follows:

$$RRA_t = -\frac{u''_c}{u'_c} c_t$$

In the *laissez-faire* equilibrium, this average level is 32. Once the tax is introduced, this falls to around 13. The main effect of the tax is then to increase the distance between consumption and the externality. As in Campbell and Cochrane (1999), risk aversion in our model is determined by "surplus consumption". A greater distance between consumption and the externality therefore implies a lower coefficient of relative risk aversion.

5.7 Climate policy and asset prices with standard preferences

In many models, the EIS mainly affects quantities, whereas asset-pricing implications are driven by risk aversion (e.g. Cochrane 2017; Tallarini 2000). In contrast, the financial and macroeconomic implications of our model are tightly linked. The specification with CCN preferences creates this interaction between finance and the environmental policy. This point is illustrated in Table 8, which repeats the experiment shown in Table 3 using a separable specification. We analyze the effect of the optimal policy in a model in which preferences are as follows:

$$\mathcal{W}_t = E_0 \sum_{t=0}^{\infty} \widetilde{\beta}^t \left(\log c_t - \phi \frac{x_t^{\chi}}{\chi} \right)$$

where, following Stokey (1998), χ is set to 1.2. To ensure comparability, the parameter ϕ is calibrated to imply an optimal tax similar to that obtained in the case of CCN preferences.

With constant relative risk-aversion, the model is no longer able to generate a realistic risk premium in the *laissez-faire* equilibrium. Relative to the case of CCN preferences, the risk premium falls from about 1.2 percent to essentially 0. In this case, the dichotomy between climate policies and finance is also close to perfect. Indeed, as illustrated in Table 8 the introduction of the optimal tax essentially has no effect on the risk-free rate and risk premium. In a model in which risk plays no role, one may therefore be tempted to conclude that climate risk and environmental policies have a negligible effect on financial markets.

Since we use a log utility specification for consumption, we also tried to increase the curvature coefficient from 1 to 20. We find that increasing curvature has a negligible impact on the risk premium but generates a very large increase in the mean risk-free rate. With a high curvature coefficient, the optimal policy also has no effect on the model's asset-pricing implications. Therefore, the dichotomy between climate policies and finance cannot be broken by a very high value of the curvature coefficient.

5.8 The responses to shocks

Figure 4 compares the response of consumption c, abatement μ, emissions e and the optimal tax τ following a positive technology shock. As can be seen by comparing the red crosses to the green circles in the upper-left panel, the first key difference is that the response of consumption on impact is stronger under the optimal policy. This can be explained by the lower EIS. In models with habits, relative risk aversion and the EIS are connected. As the tax reduces risk aversion, it also increases the EIS.

As illustrated in the upper-right panel of Figure 4, the second key difference is that the quantity of emissions that firms choose to abate increases sharply during boom periods. Once the optimal policy is introduced, firms therefore find it optimal to use the abatement technology to reduce the burden of the tax.

The lower left panel of Figure 4 shows that the pro-cyclical response of the abatement technology implies lower emissions under the optimal policy. In contrast to the *laissez-faire* equilibrium, emissions therefore become counter-cyclical once the optimal tax is introduced.

Finally, the lower-right panel of Figure 4 depicts the response of the optimal tax, which is constant and equal to zero in the *laissez-faire* equilibrium. As in Ljungqvist and Uhlig (2000), the optimal tax is pro-cyclical when the economy is hit by a technology shock. Relative to the decentralized equilibrium, the planner therefore chooses to cool down the economy during booms.

The response to an investment-specific technology shock is shown in Figure 5. This shock generates a negative co-movement between consumption and investment. Relative to the *laissez-faire equilibrium*, the optimal policy attenuates the fall in investment by reducing the tax as well as abatement. Introducing this shock reduces the volatility of investment, which in turn explains the lower value of the adjustment-cost parameter that we find compared to Jermann (1998).

The response to a government spending shock is shown in Figure 6. In both cases, a positive government-spending shock reduces consumption. In our model, this can first be explained by the negative wealth effect from the shock. On impact, the shock has no effect on production, but increases the share of output allocated to government spending. On impact, consumption and investment therefore have to fall.

This negative wealth effect is reinforced by a negative substitution effect. As in models with habits and adjustment costs, this reflects the increase in the real interest rate generated by the shock. As agents become more reluctant to save as consumption falls, the real interest rate has to rise to restore equilibrium.

This illustrates the trade-off between environmental protection and macroeconomic stabilization in this model. Whereas emissions decline in the *laissez-faire* case, the social planner chooses to increase the stock of pollution. The social planner internalizes that the shock reduces the resources available for consumption. It is therefore optimal to mitigate the effect of the shock by lowering abatement as well as the tax (see the upper-right and lower-right panels of Figure 6). When the consumption cost is too large, environmental policy is used to mitigate the adverse effect of the shock. In this case, the planner chooses macroeconomic stabilization over environmental protection.

Relative to a standard business-cycle model, the main innovation is the introduction of emission shocks. In the *laissez-faire* equilibrium, consumption falls on impact and then increases above its steady-state level (see the upper-left panel of Figure 7). As emission shocks do not affect output, their main effect is to reduce "surplus consumption". The only way to mitigate the effect of this rise in the emissions stock is then to increase consumption. The problem is that to do so income has to rise first. The only way of raising income in this model is to accumulate capital. This explains why on impact consumption needs to fall. This fall is necessary to finance an increase in investment, which in turn allows agents to increase output. A few quarters after the shock, as the higher investment raises output, consumption gradually increases. The short-term decline in consumption is therefore compensated by a rise in the medium-term. As illustrated by the red dotted line in the upper-left panel of Figure 7, consumption initially declines and then increases above its steady state a few periods after the shock.

As can be seen by comparing the red-dotted and green-circled line, the response of consumption and emissions is very different under the optimal policy. The planner chooses to allocate a large fraction of resources to the abatement technology. It is therefore optimal to reduce consumption and investment to finance abatement to prevent emissions from rising.

As illustrated in the lower-right panel, the social planner also chooses to reduce the tax. The tax reduction helps to mitigate the fall in consumption and investment that is necessary to finance abatement.

6 Robustness checks

This section discusses two robustness checks. First, asset-pricing models are not only evaluated in terms of their ability to match asset market facts. Reproducing the volatility of macroeconomic aggregates, such as consumption, is also an important test for this class of models. Second, since we use a solution method that is relatively novel, we compare it to other nonlinear methods that are more widely-used in the literature.

6.1 The volatility of consumption

As discussed in subsection 4.4, the model overstates the volatility of consumption when simulated. Using consumption as an observable variable ensures that the model can perfectly reproduce the historical path of consumption growth over the estimation period. However, when simulated using the estimated values for the shock parameters, and as shown in Table 1, we obtain that consumption is more volatile than output, which does not fit the facts. This naturally raises the concern that our model's ability to generate realistic asset-pricing facts comes at the cost of implausibly-large fluctuations in consumption growth.

This section shows that this counterfactual implication does not affect the main message of the paper. To illustrate, we consider a simplified version of the model in Section 2 in which technology shocks are the only source of business-cycle fluctuations and where all variables grow at the same rate. Then, following the analysis in Jermann (1998), we calibrate the main model parameters to maximize its ability to match a set of moments that includes the volatility of consumption.

To ease the comparison with Jermann (1998), we target the same stylized facts, with one exception, and calibrate a similar set of parameters using the simulated method of moments. In our case, the five parameters are: (i) the adjustment-cost parameter, ϵ; (ii) the marginal-damage parameter, ϕ; (iii) the subjective discount factor, β; (iv) the technology-shock standard deviation, σ_A; and (v) the shock-persistence parameter, ρ_A. The first four moments to match are the standard deviations of output, consumption and investment, and the mean risk-free rate. Since the model in Jermann (1998) tends to generate excessive risk-free rate variations, we target a risk-free rate standard deviation of 5 percent instead of a 6.18 percent risk premium. The loss function is minimized for the following combination of parameter values:

ϵ	ϕ	β^Y	σ_A	ρ_A
0.36	0.0028	0.993	0.01	0.96

All other parameter values are kept at their estimated values. The moments corresponding to the laissez-faire economy appear in the first column of Table 4. Compared to Jermann (1998), the model generates a lower risk-free rate standard deviation and is still able to reproduce the low mean risk-free rate as well as the volatility of macroeconomic aggregates. As regards the moments that were not targeted, shown in the last two rows of Table 4, the model generates a bond premium of 3.4 percent. As the carbon tax is zero in the laissez-faire economy, the abatement chosen by firms is constant at a value of zero.

The second column of Table 4 lists the simulated moments when the optimal tax is introduced. As in the previous section, we first consider a scenario in which firms are able to abate around 50 percent of all emissions under the tax. The moments in this scenario appear under the column $\mu = 0.5$. Comparing the laissez-faire economy to the optimal-tax case, the risk-free rate rises by about one percentage point, and the risk premium falls under the optimal policy. The effect on the risk premium is particularly large, as the tax generates a fall of about 2.4 percentage points. Moreover, relative to the analysis from the previous section, this sizeable effect is obtained in a model with one single source of shocks. To sum up, the optimal tax also has sizeable asset-pricing implications in a version of the model that reproduces the fact that consumption is half as volatile as output.

The second main takeaway is that this robustness analysis confirms the importance of the abatement technology in our results. If firms can only abate 10 percent of emissions following the tax, the effect on the risk-free rate and the bond premium becomes negligible. This scenario corresponds to the case with an inefficient abatement technology.

This result also confirms that the asset-pricing effect that we obtain depends critically on the additional margin that is activated by the optimal policy. Once the optimal en-

	Laissez-faire Economy	Optimal Policy	
	$\mu = 0.0$	$\mu = 0.5$	$\mu = 0.1$
$std(y)$	1.0	1.0	1.0
$std(c)$	0.5	0.3	0.8
$std(i)$	2.5	0.7	1.5
$400E(r_t^F)$	0.9	1.9	0.8
$400std(r_t^F)$	5.2	2.3	4.8
Non-targeted moments			
$400E(r_{t+1}^B - r_t^F)$	3.4	1.0	3.2
$E(\mu)$	0	0.5	0.1

Table 4: Laissez-faire vs. Optimal Policy

vironmental tax is introduced, the abatement technology is used to reduce the amount of consumption risk in the economy. If sufficiently flexible, this margin helps agents to smooth consumption, which in this class of models not only reduces precautionary savings but also the compensation for holding a risky asset such as a long-term bond.

6.2 Comparison with the particle filter

In this section, we investigate whether our results continue to hold with alternative filtering methods other than the inversion filter. In the asset-pricing literature, the natural benchmark for non-linear models is particle filtering, as the latter allows likelihood-based inference of nonlinear and/or non-normal macroeconomic models (e.g. van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez, 2012; Andreasen, 2012). The inversion and particle filters are algorithms that recursively update and estimate the state and find the innovations driving a stochastic process given a set of observations.

The inversion filter does so by inverting the model's recursion rule, while the particle filter uses a sequential Monte Carlo method. Both estimation methods require the use of numerical approximation techniques that introduce error between the "true" value of the parameter and its estimate.

In the implementation of the particle filter, it is common to posit that the data-generating process (DGP) includes measurement errors. As underlined by Cuba-Borda et al. (2019), the presence of measurement error may seem to be an innocuous way of getting around degeneracy issues when choosing a computationally-manageable number of particles. As the number of innovations must be the same as the number of observables, the inversion filter may exhibit misspecification errors if measurement errors are part of the DGP. It is nonetheless standard to assume no measurement errors for linearized models, following Smets and Wouters (2007).

Sample:	Historical Data		Artificial Data
Filter:	(1) Particle	(2) Inversion	(3) Inversion
ESTIMATED PARAMETERS			
Productivity AR(1)	0.9714 *[0.9459;0.9851]*	0.9727	0.9632
Productivity std	0.0074 *[0.0067;0.0080]*	0.0076	0.0075
PREMIUM			
Premium laissez-faire	0.7500 *[0.6230;0.9118]*	0.8412	0.7867
Premium tax policy	0.3516 *[0.2851;0.4232]*	0.3774	0.3759

Notes: 25,000 iterations of the random-walk Metropolis-Hastings algorithm are drawn for the posterior uncertainty for each model. The maximization of the mode is carried out via simplex optimization routines. The confidence intervals in column(1) are drawn from the posterior uncertainty from 1,000 draws from the Metropolis-Hastings algorithm. The artificial data in column (1) are obtained from 1,000 simulations of the estimated model with the particle-filtering method.

Table 5: Outcomes from the particle vs. inversion filters under historical and simulated data

To gauge how much our results are robust to misspecification errors, we estimate our model solved up to the second order with innovations to productivity estimated with output growth as an observable variable. We limit ourselves to productivity shocks as these are the main driver of the risk premium. The rest of the parameters are set to the posterior mean taken from the previous estimation in Table 7. We consider three situations: (1) the particle filter algorithm as described in Fernández-Villaverde and Rubio-Ramírez (2007) estimated on US data;[14] (2) the inversion filter estimated on US data; and (3) the inversion filter estimated on 1,000 simulated output-growth data from the particle filter from column (1) that includes measurements error. The latter allows us to see whether measurement errors affect the inference of structural parameters when using the inversion filter. Table 5 shows the results.

[14]We use 10,000 particles to approximate the likelihood, and set the variance of the measurement errors to 10% of the sample variance of the observables to help estimation. These values are very standard in the literature.

The comparison of columns (1) and (2) shows whether the inversion filter and particle filter outcomes differ. The two filters provide a very similar measure of the likelihood function, as the differences in the inference of structural parameters are only minor. In particular, the outcome from the inversion filter always lies in the confidence interval of that from the particle filter, both for the estimated structural parameters and the premium effects. The fact that the lower risk premium from environmental policy is very similar across estimation methods is also reassuring, and suggests that our results may remain similar under alternative filtering methods.

To make sure that the robustness of our results to measurement errors holds unconditionally in larger samples, we follow Fernández-Villaverde and Rubio-Ramírez (2005) and simulate 1,000 output-growth data from the model in column (1). We estimate the model on this artificial data using the inversion filter and list the outcomes in column (3). The inversion filter infers a value that is close to the true parameter values, despite the presence of measurement errors.

7 Conclusion

Drawing from the macroeconomic, financial, and environmental literatures, this paper introduces an environmental externality into the neoclassical growth model. Our first main takeaway is that the optimal carbon tax is determined by the implicit price of CO2 emissions. We then show how to use asset-pricing theory to estimate the optimal carbon tax over the business cycle.

In our economy, risk aversion is higher when firms do not internalize the damage caused by emissions. We show that this higher risk aversion in turn raises risk premia and lowers the natural rate of interest by increasing precautionary saving. In the *laissez-faire* equilibrium, the key is that a fraction of these variations in risk aversion are excessive. The optimal policy therefore eliminates inefficient fluctuations in risk aversion.

The main policy implication is that the effectiveness of the policy critically depends on the abatement technology, so that policy success may depend on the timing of implementation. Clearly, improving the existing emission-abatement technology should come first. Once available, an efficient technology would help to mitigate the side effects of the tax, thereby maximizing the welfare gains from the policy.

As our study focuses primarily on tax policy, future research could investigate how a permits market could affect asset prices and welfare, either by considering the case of asymmetric information,[15] or by developing a framework where both households and firms are affected by the externality. This type of framework would allow for multi-policy evaluation, such as the comparison of tax and cap-and-trade policies.

Another important limitation of our analysis is that the deterministic growth rate of the economy is given exogenously. On the contrary, abatement choice is endogenously determined, and as we are primarily interested in the cyclicality of the carbon tax, our analysis focuses on business-cycle frequency. Addressing this question in a unified framework in which long-term growth and business cycle fluctuations can be jointly analyzed would be a major step forward.

We also restrict our analysis to the case of fiscal policy, and do not study the interaction between the carbon tax and other policy instruments. Understanding how the optimal carbon tax will affect the conduct of monetary and macro-prudential policies is another important avenue for further research (e.g. Benmir and Roman, 2020).

[15] Asymmetric information breaks the equivalence between the tax and the permit policy (Heutel 2012).

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9 Appendix - A: tables

Model counterpart	Name	Values
$\bar{N}$	Labor supply	0.20
δ_K	Depreciation rate of capital	0.025
$\bar{g}/\bar{y}$	Public spending share in output	0.20
$\bar{x}$	Atmospheric carbon (gigatons) in laissez-faire	800
$[4(1-\gamma_A\gamma_E^{1-\varphi_2}\eta)]^{-1}$	Half-life of CO2 in years	139
θ_1	Abatement cost	0.05607
θ_2	Curvature abatement cost	2.8

Table 6: Calibrated parameter values (Quarterly basis)

		Prior distributions			Posterior distributions
		Shape	Mean	Std.	Mean *[0.050;0.950]*
Shock processes:					
Std. productivity	σ_A	$\mathcal{IG}_1$	0.01	1	0.008 *[0.007;0.009]*
Std. spending	σ_G	$\mathcal{IG}_1$	0.01	1	0.035 *[0.032;0.039]*
Std. abatement	σ_X	$\mathcal{IG}_1$	0.01	1	0.020 *[0.018;0.022]*
Std. investment	σ_I	$\mathcal{IG}_1$	0.01	1	0.014 *[0.012;0.016]*
AR(1) productivity	ρ_A	$\mathcal{B}$	0.50	0.20	0.944 *[0.930;0.955]*
AR(1) spending	ρ_G	$\mathcal{B}$	0.50	0.20	0.953 *[0.932;0.967]*
AR(1) abatement	ρ_X	$\mathcal{B}$	0.50	0.20	0.896 *[0.828;0.947]*
AR(1) investment	ρ_I	$\mathcal{B}$	0.50	0.20	0.998 *[0.998;0.999]*
Structural parameters:					
Productivity growth rate	$(\gamma_A - 1) \times 100$	$\mathcal{G}$	0.50	0.04	0.340 *[0.301;0.387]*
Output-CO2 (de)coupling rate	$(\gamma_E - 1) \times 100$	$\mathcal{N}$	0	0.25	-0.45 *[-0.538;-0.346]*
Discount rate	$(\beta^{-1} - 1) \times 100$	$\mathcal{G}$	0.50	0.25	0.139 *[0.051;0.343]*
Capital intensity	α	$\mathcal{B}$	0.25	0.02	0.412 *[0.374;0.453]*
Capital-cost elasticity	ϵ	$\mathcal{G}$	4	1	1.448 *[1.029;2.038]*
Utility loss on emissions	$\phi \times \bar{c}/\bar{x}$	$\mathcal{U}$	0.50	0.285	0.677 *[0.611;0.730]*
Relative risk aversion	σ	$\mathcal{G}$	2.00	0.35	4.198 *[3.681;4.740]*
Output-CO2 elasticity	φ_2	$\mathcal{B}$	0.50	0.20	0.367 *[0.138;0.633]*
Log-marginal data density					-2124.0769

Notes: $\mathcal{B}$ denotes the Beta, $\mathcal{IG}_1$ the Inverse Gamma (type 1), $\mathcal{N}$ the Normal, and $\mathcal{U}$ the uniform distribution.

Table 7: Prior and Posterior distributions of structural parameters

The log utiliy case ($u = \log c_t - \phi \frac{x_t^{\chi}}{\chi}$)

	Laissez-faire	Optimal policy		
		$\theta_1 = 0.05607$	$\theta_1 = 0.48164$	$\theta_1 = 6.4039$
	(1)	(2)	(3)	(4)
Business-cycle variables				
$E(c_t)$	0.5274	0.5136	0.5178	0.5210
$E(x_t)$	804.3029	348.5493	629.2131	745.8746
$E(\mathcal{W}_t)$	-1102.7147	-673.4293	-921.8315	-1043.4189
$E(\tau_t)$	0.0000	0.0389	0.0530	0.0581
Asset-pricing implications				
$400E\left(r_t^F\right)$	6.2415	6.2430	6.2425	6.2421
$400E\left(r_{t+1}^B - r_t^F\right)$	0.0715	0.0696	0.0704	0.0709
$std(\hat{\lambda}_t)$	0.2194	0.2135	0.2130	0.2134
$E(RRA_t)$	1.0000	1.0000	1.0000	1.0000
$std(\widehat{rra}_t)$	0.0000	0.0000	0.0000	0.0000
Abatement technology				
$E(\mu_t)$	0.0000	0.5563	0.1999	0.0500
$E(f(\mu_t))$	0.0000	0.0111	0.0055	0.0015
$E(\frac{\tau_t e_t}{y_t})$	0.0000	0.0239	0.0597	0.0781

Notes: The first column shows the results in the laissez-faire (counter-factual) equilibrium, where we use the estimated values obtained for non-separable utility. We calibrate $\phi = 5.7105e-05$ in order to match the optimal tax obtained in the case of non-separable utility. Column (2) is the equilibrium under an environmental tax with θ_1 set as in the literature. Columns (3) and (4) are equilibria under alternative values of θ_1 that match abatement shares of $\bar{\mu}$ of 20% and 5%. Note that $E(\mu_t) \neq \bar{\mu}$ in columns (3) and (4) due to the contribution of future shocks to the asymptotic mean of these variables.

Table 8: Counter factual robustness check – The case of separable utility.

10 Appendix - B: figures

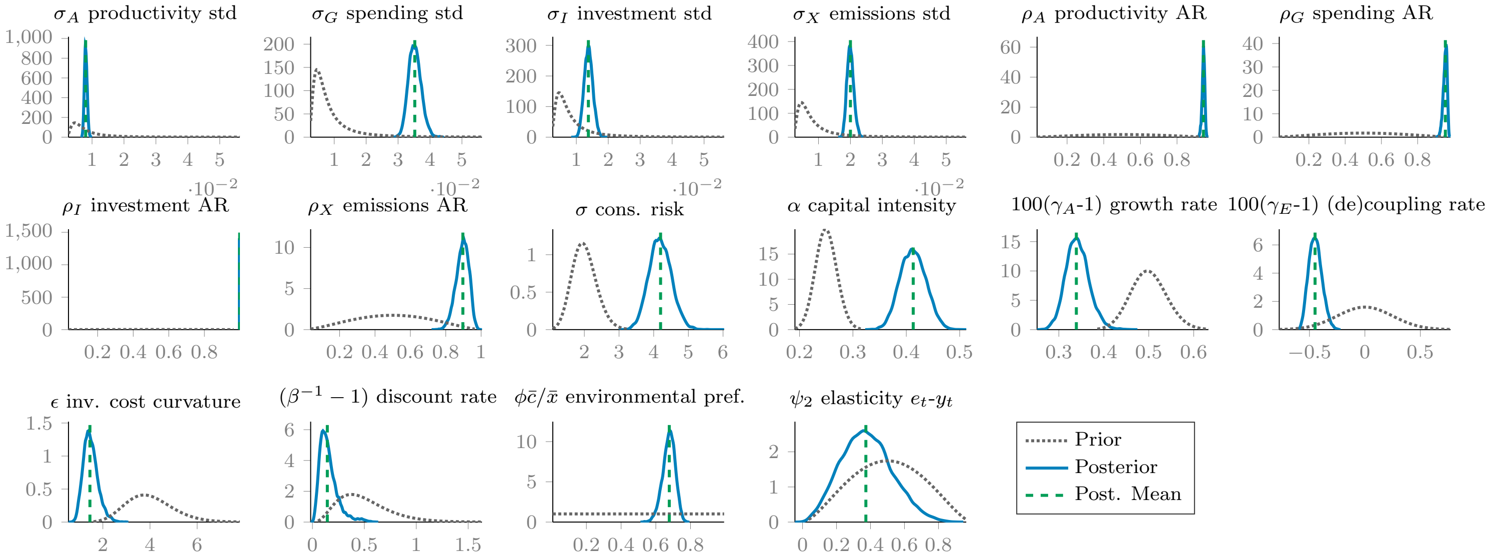
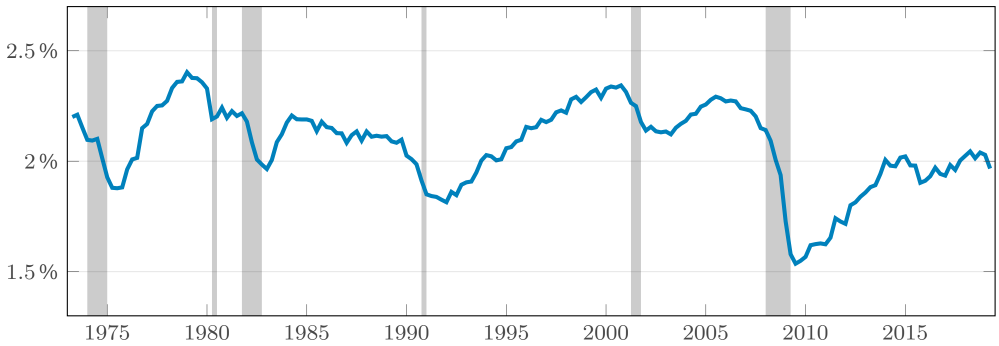
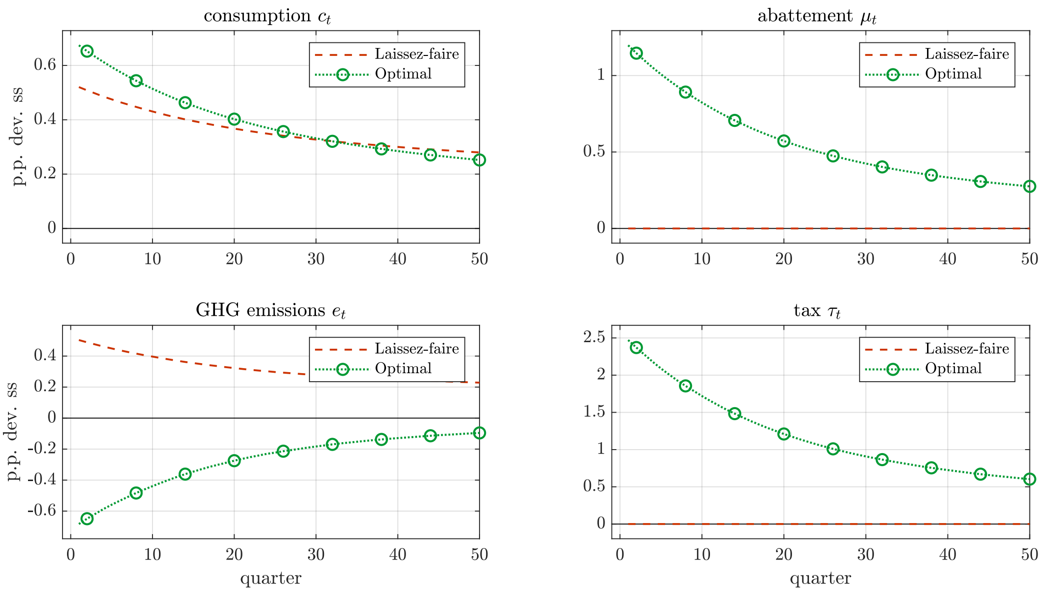


Figure 2: Prior and posterior distributions of the estimated parameters



Notes: The simulated path is expressed in levels. The blue line represents the mean of 1,000 simulated paths of Metropolis-Hastings random iterations. The gray shaded areas are NBER-dated recessions in the US.

Figure 3: Historical variations in the tax bill in % of GDP, $\tau_t e_t / y_t$



Notes: The IRFs are generated using a second-order approximation to the policy function and are expressed as percentage deviations from the deterministic steady state. Estimated parameters are taken at their posterior mean.

Figure 4: Impulse responses from an estimated TFP shock

Figure 5: Impulse responses from an investment-specific technology shock

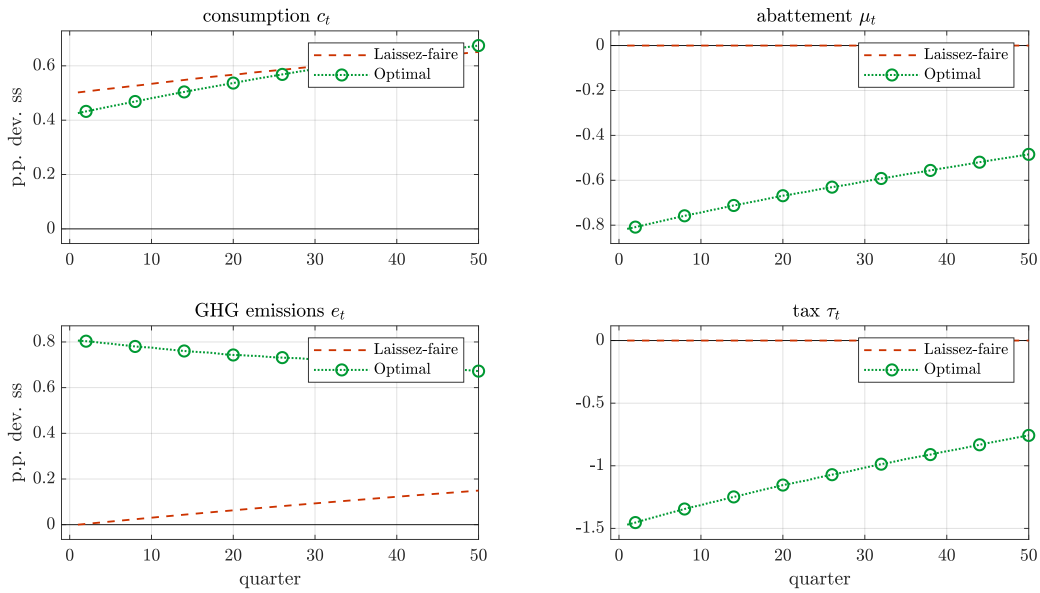
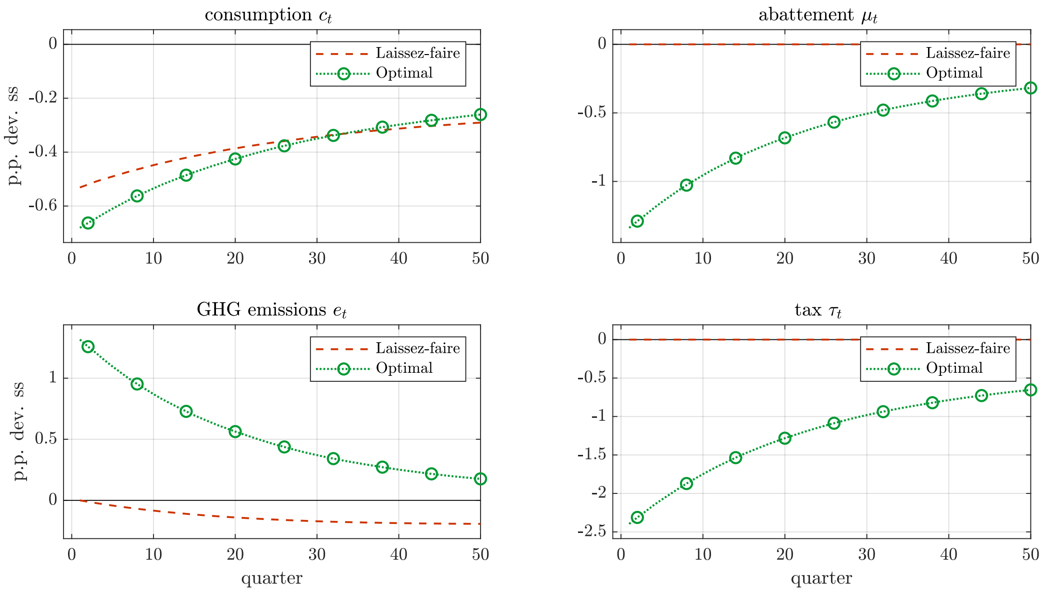


Figure 6: Impulse responses from a government-spending shock



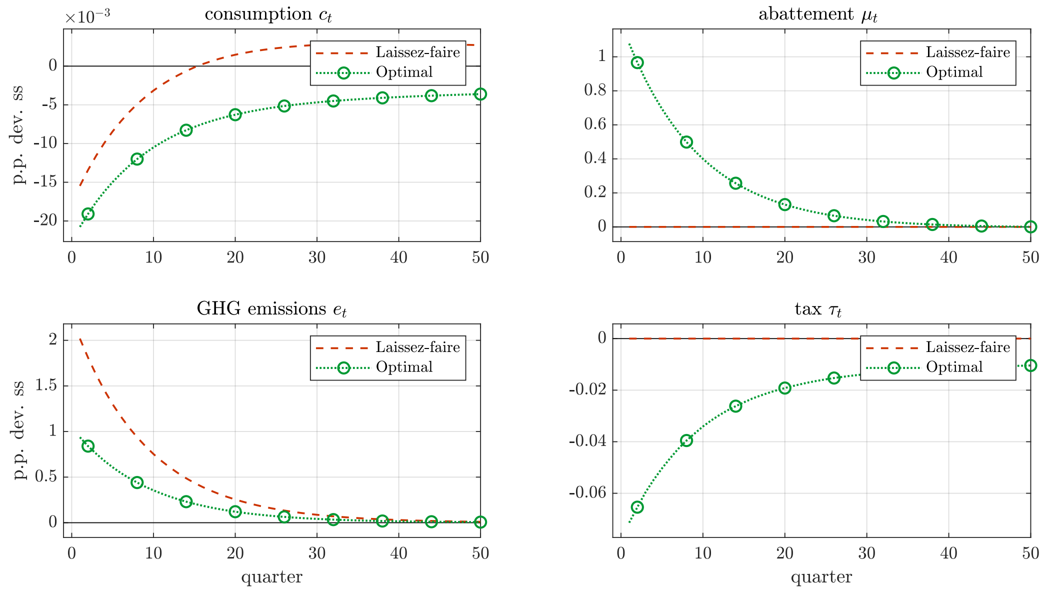


Figure 7: Impulse responses from an emissions shock

11 Appendix - C: Balanced growth (not for publication)

Labor-augmenting technological progress is denoted by Γ_t. The growth rate of Γ_t determines the growth rate of the economy along the balanced growth path. This growth rate is denoted by γ^Y, where:

$$\Gamma_{t+1} = \gamma^Y \Gamma_t \tag{25}$$

Stationary variables are denoted by small caps, whereas variables that are growing are denoted by capital letters. For example, in the growing economy output is denoted by Y_t. De-trended output is thus obtained by dividing output in the growing economy by the level of labor-augmenting technological progress:

$$y_t = \frac{Y_t}{\Gamma_t} \tag{26}$$

The production function of emissions is also subject to technological progress. We denote the level of Green technological progress by Ψ_t. The growth rate of Green technological progress is γ^E.

$$\Psi_{t+1} = \gamma^E \Psi_t \tag{27}$$

Note that an improvement in the Green technology implies a value for γ^E that is below one.

11.1 The de-trended economy

In the growing economy, with labor-augmenting technological progress, the production function is as follows:

$$Y_t = \varepsilon_t^A K_t^{\alpha} (\Gamma_t n_t)^{1-\alpha} \tag{28}$$

where hours worked n_t and the technology shock ε_t^A are stationary variables.

In the de-trended economy, we have that:

$$y_t = \varepsilon_t^A k_t^\alpha {n_t}^{1-\alpha} \tag{29}$$

Moreover, the economy's resource constraint is:

$$y_t = c_t + i_t + f(\mu_t) y_t \tag{30}$$

where the share of abated emissions μ_t is a stationary variable between 0 and 1. The capital-accumulation equation in the growing economy is:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{31}$$

In the de-trended economy, we thus have that:

$$\gamma^Y k_{t+1} = (1-\delta)k_t + i_t \tag{32}$$

Emissions, which we denote by E_t, in the growing economy are given as follows:

$$E_t = (1-\mu_t)\varphi_1 Y_t^{1-\varphi_2} \Psi_t \tag{33}$$

where φ_1 and φ_2 are parameters.

In the de-trended economy, we have that:

$$e_t = (1-\mu_t)\varphi_1 y_t^{1-\varphi_2} \tag{34}$$

where:

$$e_t = \frac{E_t}{\Psi_t \left(\Gamma_t\right)^{1-\varphi_2}} \tag{35}$$

In the growing economy, the stock of emissions in the atmosphere is denoted by X_t. The accumulation of emissions in turn depends on the level of new emissions E_t :

$$X_{t+1} = \eta X_t + E_t \tag{36}$$

where η is the fraction of the stock of emissions that remains in the atmosphere.

In the de-trended economy, we have that:

$$\gamma^X x_{t+1} = \eta x_t + e_t \tag{37}$$

where, to simplify notation, we define γ^X as follows:

$$\gamma^X = \gamma^E \left(\gamma^Y\right)^{1-\varphi_2} \tag{38}$$

In the growing economy, the utility function is as follows:

$$\sum_{t=0}^{\infty} \beta^t \frac{\left(C_t - \Theta_t X_t\right)^{1-\sigma}}{1-\sigma} \tag{39}$$

where C_t is consumption, β the subjective discount factor, σ the curvature parameter, and Θ_t a preference parameter that measures the disutility caused by the stock of emissions.

The de-trended utility function takes the following form:

$$\sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{\left(c_t - \phi x_t\right)^{1-\sigma}}{1-\sigma} \tag{40}$$

where, to simplify notation, we define $\widetilde{\beta}$ as follows:

$$\widetilde{\beta} = \beta \gamma^{1-\sigma} \tag{41}$$

A stationary utility function is obtained by assuming that the preference parameter Θ_t has a trend. In the de-trended economy, the preference parameter is constant, which implies the following relationship between Θ_t and ϕ.

$$\Theta_t = \phi \frac{\left(\Gamma_t\right)^{\varphi_2}}{\Psi_t} \tag{42}$$

12 Appendix - D: The optimal tax (not for publication)

12.1 Centralized problem

We characterize here the first-best equilibrium. A social planner maximizes welfare, which leads producers to internalize the social cost of emissions. The problem for the social planner reads as follows:

$$
\begin{aligned}
\mathcal{L} = E_0 & \left\{ \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{(c_t - \phi x_t)^{1-\sigma}}{1-\sigma} \right. \\
& + \sum_{t=0}^{\infty} \widetilde{\beta}^t \lambda_t \left[y_t - c_t - i_t - g_t - f(\mu_t) y_t \right] \\
& + \sum_{t=0}^{\infty} \widetilde{\beta}^t \lambda_t q_t \left[(1-\delta) k_t + \left[\frac{\chi_1}{1-\epsilon} \left(\varepsilon_t^I \frac{i_t}{k_t} \right)^{1-\epsilon} + \chi_2 \right] k_t - \gamma^Y k_{t+1} \right] \\
& + \sum_{t=0}^{\infty} \widetilde{\beta}^t \lambda_t \varrho_t \left[\varepsilon_t^A k_t^{\alpha} n_t^{1-\alpha} - y_t \right] \\
& + \sum_{t=0}^{\infty} \widetilde{\beta}^t \lambda_t v_{Xt} \left[\gamma^X x_{t+1} - \eta x_t - e_t \right] \\
& \left. + \sum_{t=0}^{\infty} \widetilde{\beta}^t \lambda_t v_{Et} \left[e_t - (1-\mu_t) \varepsilon_t^X \varphi_1 y_t^{1-\varphi_2} \right] \right\}
\end{aligned}
$$

The marginal utility of consumption c_t is:

$$
\lambda_t = (c_t - \phi x_t)^{-\sigma} \tag{43}
$$

Optimal investment i_t is given by:

$$
1 = \varepsilon_t^I q_t \chi_1 \left(\varepsilon_t^I \frac{i_t}{k_t} \right)^{-\epsilon} \tag{44}
$$

The optimal capital supply is given by:

$$
q_t = \beta^Y E_t \frac{\lambda_{t+1}}{\lambda_t} \left\{ q_{t+1} \left((1-\delta_K) + \frac{\chi_1}{1-\epsilon} \left(\varepsilon_{t+1}^I \frac{i_{t+1}}{k_{t+1}} \right)^{1-\epsilon} + \chi_2 - \chi_1 \left(\varepsilon_{t+1}^I \frac{i_{t+1}}{k_{t+1}} \right)^{1-\epsilon} \right) + \varrho_{t+1} \alpha \frac{y_{t+1}}{k_{t+1}} \right\}
$$

where:

$$
\beta^Y = \widetilde{\beta} / \gamma^Y
$$

The first-order condition on output y_t is:

$$
[1 - f(\mu_t)] - \varrho_t - v_{Et} (1-\varphi_2) \frac{e_t}{y_t} = 0
$$

The optimal fraction of abatement μ_t is given by:

$$f'(\mu_t)\,y_t = v_{Et}\frac{e_t}{(1-\mu_t)} = 0 \tag{45}$$

The optimal quantity of emissions e_t per quarter reads as follows:

$$v_{Et} = v_{Xt} \tag{46}$$

While the shadow value of pollution is:

$$\lambda_t v_{Xt} = \beta^X E_t \phi \left(c_{t+1} - \phi x_{t+1}\right)^{-\sigma} + \eta \beta^X E_t \lambda_{t+1} v_{Xt+1} \tag{47}$$

where:

$$\beta^X = \widetilde{\beta}/\gamma^X \tag{48}$$

12.2 Laissez-faire equilibrium

Assume the following functional form for $f(\mu_t)$:

$$f(\mu_t) = \theta_1 \mu_t^{\theta_2} \tag{49}$$

Firms are profit-maximizing:

$$\max_{k_t, n_t, \mu_t, e_t} d_t = y_t - w_t n_t - i_t - \theta_1 \mu_t^{\theta_2} y_t - \tau_t e_t$$

Subject to the capital-accumulation constraint:

$$\gamma^Y k_{t+1} = (1-\delta)k_t + \left(\frac{\chi_1}{1-\epsilon}\left(\varepsilon_{It}\frac{i_t}{k_t}\right)^{1-\epsilon} + \chi_2\right)k_t \tag{50}$$

Subject to the emission law of motion:

$$e_t = \varepsilon_{Xt}(1-\mu_t)\varphi_1 y_t^{1-\varphi_2} \tag{51}$$

And subject to the supply curve:

$$y_t = \varepsilon_{At} k_t^{\alpha} n^{1-\alpha} \tag{52}$$

The Lagrangian reads as follows:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{\lambda_t}{\lambda_0} \left\{ \begin{array}{c} y_t - w_t n - i_t - \theta_1 \mu_t^{\theta_2} y_t - \tau_t e_t \\ + v_{Et} \left[e_t - \varepsilon_{Xt}(1-\mu_t)\varphi_1 y_t^{1-\varphi_2} \right] \\ + \varrho_t \left[\varepsilon_{At} k_t^{\alpha} n^{1-\alpha} - y_t \right] \\ + q_t \left[(1-\delta) k_t + \left(\frac{\chi_1}{1-\epsilon} \left(\varepsilon_{It} \frac{i_t}{k_t} \right)^{1-\epsilon} + \chi_2 \right) k_t - \gamma^Y k_{t+1} \right] \end{array} \right\}$$

The first-order condition on emissions e_t is given by:

$$v_{Et} = \tau_t \tag{53}$$

Optimal minimization of labor inputs N_t reads as:

$$w_t = \varrho_t (1-\alpha) \frac{y_t}{n_t} \tag{54}$$

The optimal quantity of physical capital k_{t+1}:

$$\lambda_t q_t = \beta^Y E_t \lambda_{t+1} q_{t+1} \left[(1-\delta_K) + \frac{\chi_1}{1-\epsilon} \left(\varepsilon_{It+1} \frac{i_{t+1}}{k_{t+1}} \right)^{1-\epsilon} + \chi_2 - \chi_1 \left(\varepsilon_{It+1} \frac{i_{t+1}}{k_{t+1}} \right)^{1-\epsilon} \right] + \beta^Y E_t \lambda_{t+1} \alpha \frac{y_{t+1}}{k_{t+1}} \varrho_{t+1}$$

The marginal profit for an additional unit produced is:

$$\varrho_t = 1 - \theta_1 \mu_t^{\theta_2} - v_{Et}(1-\varphi_2) \frac{e_t}{y_t} \tag{55}$$

Optimal abatement μ_t is given by:

$$v_{Et} \frac{e_t}{1-\mu_t} = \theta_1 \theta_2 \mu_t^{\theta_2 - 1} y_t \tag{56}$$

In the laissez-faire economy, there is no environmental policy:

$$\tau_t = 0$$

Recall that firms do not consider the stock of emissions x_t as a state variable. In equilibrium the cost of carbon v_{Xt}, as considered by firms, is 0 because they do not internalize the effects of emissions on households. As a result, since in the laissez-faire equilibrium τ_t is set to 0, the first-order conditions with respect to emissions imply that $v_{Et} = 0$. From the first-order conditions with respect to μ_t and y_t, this in turn implies $\mu_t = 0$ and $\varrho_t = 1$.

12.3 Competitive equilibrium under optimal policy

The first-best equilibrium that corresponds to the problem of the social planner can be attained by setting the tax τ_t equal to the price of carbon. In the centralized equilibrium, the price of carbon is determined by the optimality condition with respect to x_t. The optimal tax is therefore:

$$\tau_t = v_{Xt} \tag{57}$$

Once the optimal tax is implemented, in the laissez-faire equilibrium, equation (53) then implies that:

$$v_{Et} = v_{Xt} \tag{58}$$

The optimality condition shown in equation (46) is therefore satisfied, as the cost of abating emissions is exactly equal to the social cost of emissions.

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