

Dynamic Present Values and the Intertemporal CAPM

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Abstract

Merton's (1973) intertemporal Capital Asset Pricing Model is difficult to reconcile with present value computation when investment opportunities are random and time-varying. Merton assumes a geometric price process, and his equilibrium is constructed by solving for the instantaneous expected rate of return. We argue that this method is generally inconsistent with present value computations. We derive an alternative version of the Intertemporal CAPM based explicitly on present value computation with a strictly exogenous cash flow. Our model is derived without restrictive assumptions about the structure of the equilibrium price process. An essential feature of our model is that prices respond endogenously to shocks in expected return. In an example model, we show that shocks to stochastic volatility is negatively correlated with stock returns in equilibrium. The correlation increases in absolute magnitude with risk aversion.

Keywords: ICAPM, Conditional CAPM, dynamic consistency, present value, dynamic equilibrium, stochastic volatility

JEL codes: G1, G12.

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1 Introduction

One of the most fundamental concepts in finance is that the price of an asset equals the present values of the assets' future cash flows. If the price of an asset differs from its present value, the price is generally inconsistent with equilibrium, and may create arbitrage opportunities. For assets with well-defined future cash flows such as bonds, it is obvious how to construct asset pricing models based on present values. However, for assets whose future payoffs are much harder to model, as is the case with common stocks, the present value approach is less straightforward. Indeed, in many asset pricing models, the financial market equilibria are often implemented by *assuming* an exogenous price process and then solving for the expected rate of return period by period. An example of this approach to constructing an equilibrium is found in Merton's (1973) seminal paper on the Intertemporal CAPM, which starts with the assumption that the price process follows a Geometric Brownian Motion with time-varying expected rates of return and volatility. The equilibrium is constructed by solving for the expected rate of return, or drift rate, of the risky price process, which in equilibrium is a function of systematic risk.

We make two contributions in this paper. First, we critique the idea that dynamic equilibrium models can be constructed simply by solving for the expected rate of return, period by period, as in Merton (1973). Second, we derive a new CAPM model, which we call the "Present-Value CAPM". This model is based on the same general assumptions as the traditional one-period CAPM except that it is derived through dynamic present value computations with time-varying discount factors.

Our critique of Merton's model is based on the fact that Merton derives his model under the assumption of an exogenously specified price process. The problem with his analysis is that the exogenous price process is not necessarily consistent with present value computation where 1) the supply of risky assets are exogenous to the model, and 2) the cash flows paid by the risky asset are strictly exogenous. The first point was demonstrated in an unpublished paper by Hellwig (1977).¹ Hellwig's analysis shows that the exogenous price assumption in Merton's model is problematic even when investors have a constant investment opportunity set. He shows by example that if investors face a normally distributed dividend stream, and exhibit constant absolute risk aversion, prices should also be normally distributed unless one is willing to impose counter-intuitive supply behaviors of risky assets - specifically, the assets' supply needs to be inversely proportional to their prices. Conversely, Hellwig demonstrates that if supplies are fixed, equilibrium prices are normally distributed, not log-normal as assumed by Merton.

¹We are grateful to Suresh Sundaresan for pointing out to us the existence of Hellwig's paper.

Our critique of Merton's ICAPM is also that the price is assumed to be exogenously given. Our argument, however, is linked to the dynamic structure of the model. The problem is that expected rates of return play two different distinct roles. Since the assets' price and future cash flows are linked together by present value computation, either the current asset price or future cash flow (or both) must adjust to reflect a change in the expected rate of return. However, the ICAPM starts with an exogenous price process that a priori assumes that the asset price does not depend on factors which impact time-varying discount rates, thus leaving no room for the current asset price to adjust simultaneously to the change in discount rates. The direct consequence of this restriction is that a corresponding change in future expected cash flow becomes the only way to implement the change in expected return. This implied change in future expected cash flow thus becomes an implicit assumption that is necessary for the price to be consistent with the present value computations. Unfortunately, as we will argue, this implicit assumption is virtually impossible to reconcile with reasonably constructed equilibria and is also at odds with casual empirical observations of the relationship between financial prices, macro-economic growth, and market volatility.

Our critique is relevant in the context of the large empirical literature on tests of the *conditional* CAPM. The conditional CAPM is a dynamic implementation of the Sharpe-Lintner-Mossin CAPM where their classic formula is assumed to hold period-by-period. The conditional CAPM has no known theoretical foundation other than Merton's (1973) continuous time ICAPM. Thus, as we question the assumptions of Merton's model, we also question the validity of the conditional CAPM. While we do not discuss the consequences for empirical tests of the intertemporal or conditional, CAPM models in detail here, it is clear that our critique has implications for tests of conditional asset pricing models. For example, the misspecification of return distributions will lead to incorrectly specified likelihood functions, which again, of course, will imply that empirical estimates of model parameters will be biased.

So how do we construct asset pricing models that would not be subject to this critique? The good news is that this is easy. In particular, pricing formulas that are derived explicitly from dividend discounting as in Gordon (1959) obviously escape our critique. Early work on dynamic present value models include Campbell and Shiller (1987), Campbell and Shiller (1988), and Campbell (1993). Recently, the long run risk (LRR) model of Bansal and Yaron (2004) endogenizes asset prices as the dynamic present value of future consumption or dividends. Part of the appeal of the LLR model is that risk premia are proportional to the inverse of the speed of mean reversion of the long run risk factor. Similarly, in a model with stochastic volatility of aggregate consumption, the volatility risk premium is proportional to the inverse of volatility speed of mean reversion. This delivers potentially very large risk premia, which in turn generate

realistically large equity returns and volatilities as well as potentially explaining other asset pricing puzzles. Cochrane (2011) discusses the dual interpretation of expected returns as a growth rate and as a discount factor. The first implies trivially that returns can be forecasted by expected returns. The latter implies a negative relationship between contemporaneous returns and shocks to discount factors. Cochrane (2011) also postulates that much of the variation in stock returns is due to changes in discount-rates themselves.

We propose a new capital asset pricing model based on the idea that we wish to derive an Intertemporal CAPM which maintains the property that prices obtain as present values. Investors in our model have utility of terminal wealth, as in the classic one period CAPM. We devise a very simple technique for constructing general, affine Present-Value based CAPM's. Our technology can be seen to generalize Merton (1973) to provide a more general price process. In particular, we do not impose any prior assumptions about the process that governs the equilibrium price.

Our model is connected to the long run risk model in some important ways. First, our present value framework yields risk premium specifications that bear close resemblance to the equivalent expressions obtained in the LRR literature. Risk premia in our model are magnified by the reciprocal of mean-reversion coefficients in a manner identical to the Intertemporal Elasticity of Substitution (IES) $\rightarrow \infty$ limit of the LRR model. In our experience, risk premia in LRR models converge very quickly as the IES increases and imply that even values of IES in the 3-5 range will yield model solutions that are close to the IES $\rightarrow \infty$ limit. Our model shares similar characteristics. Second, we do not model intermediate dividends or consumption but rather terminal wealth, so our model resembles the traditional wealth-based CAPM. This is different from LRR models at first glance. However, as we will show, when intermediate consumption is small and negligible compared to the terminal wealth, the limiting case IES $\rightarrow \infty$ in LRR model yields similar results to the wealth-based CAPM.²

While the model we develop in this paper has similarities with those in the extant literature, it does differ in some important ways. Our model is not a dividend discounting model. Since there are no intermediate dividends or consumption, we do not need to use the Campbell-Shiller linearization of the returns process to obtain closed-form expressions for stock prices. Our price formulas are exact. This is a large advantage in structural estimation of models.³

²The fact that Epstein-Zin preferences yield the wealth based CAPM as a special case when EIS $\rightarrow \infty$ is shown in the original paper by Epstein and Zin (1989), p. 968.

³In long-run-risk models it is necessary to solve a set of non-linear equations for each new parameter update in a numerical search (see Bansal, Kiku, and Yaron (2006)). Often these equations are difficult to solve for high frequency data (see Eraker (2010)).

To see how our model shares similarities and differences with existing models, consider an example. In a simple economy with a single asset where terminal payoff has stochastic volatility, we derive the equilibrium return process as

$$\ln R_{m,t+1} - r_f = \lambda_0 - \frac{\gamma}{\kappa} \Delta \sigma_t^2 + \sigma_t \epsilon_t. \quad (1)$$

The process σ_t is stochastic, stationary, and is interpretable as the volatility of beliefs about economic growth. ϵ_t is a random shock representing economic growth news. The volatility process, σ_t , is Markov, and $E_t \Delta \sigma_t^2 \approx \kappa(\theta - \sigma_t^2)$ so that $E_t(\ln R_{m,t+1}) \approx r_f + \gamma \sigma_t^2 + \text{constant}$ where the constant term reflects a risk compensation for variation in future expected cash flows as well as volatility risk. The term $-\gamma/\kappa \Delta \sigma_t^2$ is an equilibrium volatility feedback term. This volatility feedback term will lead to temporary price adjustments in order to reflect a higher or lower risk premium depending on whether or not volatility is above or below its long run mean. A positive shock to volatility leads to a simultaneous decrease in the price, allowing for higher future expected returns. The model does not yield Merton's ICAPM except in the special case when the volatility process is constant. Generally, both volatility shocks and shocks to future cash flows, ϵ_t , are priced risk factors.

Our Present-Value CAPM model is also important for several other reasons. One primary advantage of our modeling approach is that we recover an equilibrium model in which risk premia have long run risk interpretations, but without the explicit link to consumption dynamics on which the Bansal and Yaron model and its descendants are based. This can be useful in constructing realistic models of stock prices for the purpose of pricing derivatives. In our model the volatility feedback effect is endogenous. If used to price equity options, our model can produce a large volatility risk premium. The skewness in the return distribution under both the objective and risk-neutral measures is a function of risk aversion. This allows, among other things, the econometric identification of risk aversion from option prices as, among other things, the steepness of the volatility smile can be explicitly linked to risk aversion in our example model in equation 1.

Our model also has important implications for the empirical testing and evaluation of asset pricing models. Our present value CAPM model implies that stochastic volatility will lead to a predictable component in returns. This is standard for the ICAPM, but in our model, shocks to volatility generate temporary movements in stock prices that can be identified from the contemporaneous relation between the volatility shocks and returns. When the stock price goes down in response to an increase in volatility, it will mean-revert. The speed in which this reversion takes place is typically slow, which means that the correlation between volatility level and future

returns is low, whereas the contemporaneous correlation is strong. This has two econometric implications. First, we can better identify the aggregate market risk-return relationship better from the contemporaneous volatility feedback effect than we can from predictive regressions. Second, econometric inference on the relationship between volatility and returns can be misleading if inference is made from a model that ignores the contemporaneous relation. We investigate the implication for inference on risk aversion in an intertemporal CAPM in a companion paper.

The rest of the paper is organized as follows: in the next section we outline the implicit relationship between future cash flows and risk in the ICAPM. Section three develops our new general Present-Value CAPM for general valuation, and we discuss some specific examples. Section four offers concluding remarks. Proofs and lengthy derivations are in the appendices.

2 A Critique of Merton's ICAPM

In this section we outline our critique of Merton's ICAPM model. The problem with Merton's model is that it starts by *assuming* an equilibrium price process. The assumed price process may or may not be consistent with a reasonably defined dynamic equilibrium price. In particular, we will show that there exists specifications of exogenously specified cash flows and utilities that are impossible to reconcile with Merton's model unless one is willing to make additional unwarranted assumptions about the covariation between future cash flows and changes in expected returns.

We are not the first authors to criticize the exogenous price assumption in Merton's model. In an unpublished working paper, Hellwig (1977) questions the general validity of Merton's model. To proceed, we first briefly review Merton's 1973 paper.

2.1 Merton's ICAPM

Merton assumes that prices are driven by

$$\frac{dP_{i,t}}{P_{i,t}} = \alpha_{i,t}dt + \sigma_{i,t}dz_{i,t} \tag{2}$$

where $z_{i,t}$ is a Brownian motion. The Brownian motions driving prices are correlated, $\rho_{i,j} = \text{Corr}(dz_{i,t}, dz_{j,t})$. This notation follows that of Merton (1973), with the exception of the added t

subscripts for clarity. The processes governing the expected rates of return, α_i and the conditional stock market standard deviation, $\sigma_{i,t}$, are assumed to be stochastic processes,

$$d\alpha_{i,t} = a_i dt + b_i dq_{i,t}, \quad (3)$$

$$d\sigma_{i,t} = f_i dt + g_i dx_{i,t}. \quad (4)$$

Merton does not discuss whether the coefficients a_i, b_i, f_i and g_i are to be interpreted as constants, Ito-processes, deterministic functions of time, or functions of the levels of α_i and or σ_i .

Merton assumes that utility of agent i is given by

$$E_t \left[\int_t^{T_i} U^i(c_{s,i}, s) ds + B^i(W_{T_i,i}) \right], \quad (5)$$

where U^i is the agent i 's utility of his consumption stream $c_{s,i}$ and B^i is the Bequest utility. Merton derives the following equilibrium expected rate of return of the i th asset,

$$\alpha_{i,t} - r_t = \frac{\sigma_{i,t}(\rho_{i,M} - \rho_{i,n}\rho_{M,n})}{\sigma_{M,t}(1 - \rho_{n,M}^2)}(\alpha_{M,t} - r_t) + \frac{\sigma_{i,t}(\rho_{i,n} - \rho_{i,M}\rho_{M,n})}{\sigma_{n,t}(1 - \rho_{n,M}^2)}(\alpha_{n,t} - r_t). \quad (6)$$

Here the subscript M refers to the market portfolio while n refers to an n th asset where returns are assumed to be perfectly correlated with changes in the investment opportunity set (α, σ, r) . The coefficients $\rho_{i,j}$ are correlations between assets i and j and are assumed to be constant.

2.2 Hellwig's Critique

Hellwig (1977) criticizes Merton's ICAPM. The premise is the same premise as ours - the price process in Merton's model is assumed rather than derived fully endogenously. While we discuss the validity of Merton's ICAPM in the context of a changing investment opportunity set, Hellwig's analysis focuses specifically on the case where the investment opportunity set is constant. In this case, equation (6) reduces to the standard Sharpe-Lintner-Mossing static CAPM formula for logarithmic expected returns, α . Moreover, since by assumption α , σ , and r are constant, equation (2) implies that prices are log-normal. Hellwig points out that log-normality is inconsistent with dividend discounting if dividends follow an arithmetic process (i.e, if $dD_{i,t} = adt + bdB_{i,t}$ for constants a and b and a Brownian motion B_i) and investors have exponential utility, and if assets are in fixed supply. Hellwig writes, p. 1

Merton's analysis of capital market equilibrium is unsatisfactory: he determines equilibrium prices by assumption rather than by demand and supply. First, he derives the portfolio behavior under the assumption that asset prices follow a log-normal process. Then he considers the implication of market clearing for average rates of return *under a log normal process*. He fails to verify that the assumption of log-normality itself is compatible with market clearing.

Hellwig further shows that by relaxing the log-normality assumption on asset prices, he can recover the CAPM with normally distributed prices and fixed supply. This, he contends, is also unsatisfactory as it conflicts with the idea of limited liability of equity.

Hellwig also considers the possibility of solving for assets' supply in order to maintain the log-normal price assumption under the same economic assumptions (i.e, constant investment opportunities). He shows that the CAPM holds in this case, but only if prices follow an arithmetic process (not geometric, log-normal). He concludes that

“... equilibrium risky assets can only be log-normally distributed, if all agents have constant relative risk aversion, *if the supply of each risky asset is inversely proportional to its price.*”

Hellwig's analysis shows that the intertemporal CAPM of Merton is problematic even in the case of constant investment opportunities. Again, the problem with Merton's analysis is that the price process is assumed rather than derived in equilibrium, and that equilibrium can be established by just solving for the conditionally expected rate of return, $\alpha_{i,t}$.

The idea that any dynamic equilibrium model can be solved just by solving for the expected rates of return under an assumed price dynamic is troublesome. The problem is that the expected rate of return serves two purposes: it is both the average growth rate of investment as well as the discount rate of future cash flows. It is not true in general that an assumed price process is consistent with a reasonably defined equilibrium if only its expected return satisfies a first order condition, as in Merton's ICAPM.

2.3 Expected Returns in Dynamic Models

In this section we outline the basic critique of the idea of solving for expected rates of return in dynamic models. We consider, without loss of generality, capital market equilibrium in the

context of a single asset market. Let us, as in Merton, assume that the price can be written as a geometric process with time-varying drift and diffusion

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dB_t \quad (7)$$

or

$$\frac{P_{t+1} - P_t}{P_t} = \mu_t + \sigma_t \epsilon_{t+1}, \quad (8)$$

where dB_t or ϵ_{t+1} are exogenous shocks using the continuous or discrete time formulations, respectively. Merton postulates that the equilibrium price process is found through solving for the drift rate μ_t in (7) or (8). An equilibrium solution is taken to be a sequence of expected returns, μ_t , that are measurable wrt. a time t information set (say \mathcal{F}_t) available to the investor.

Merton (1980) gives the expected returns on the market portfolio as

$$\mu_t = r_{f,t} + \gamma \sigma_t^2 \quad (9)$$

Empirical investigations of Merton's ICAPM, including Merton (1980), typically implement this exact specification for the conditional expected return for the market portfolio. This particular model is a special case of Merton's ICAPM when the relative risk aversion coefficient is constant and equal to γ , and the model ignores the hedging demand terms that appear due to time-variations in the investment opportunity set in Merton's model. Merton (1980), p.329, footnote 12, argues that these hedging demands can be ignored whenever the (endogenous) consumption stream is less sensitive to changes in state-variables (that affect the investment opportunity set) than to changes in wealth, or alternatively that the variance of the change in wealth is much larger than the variance of the change in the state-variables.

Our critique of Merton's ICAPM is based upon the fact that it is not possible to write down price processes as in (7) or (8) while maintaining time-varying expected returns without simultaneously making implicit assumptions about how the future expected payoffs correlate with changes in expected return. Suppose for the sake of argument that we consider the discrete time model in (8) and that the market wide volatility σ_t is stochastic. For simplicity let us assume a finite horizon, single asset, representative agent endowment economy. Assume that at the terminal date T , the representative agent receives a payoff \tilde{x} and before T , the representative agent has no consumption. We ask the simple question: What is the value of a claim to this payoff at dates $t = 0, 1, \dots, T$? We claim that it is impossible for the price process in (8) to represent an equilibrium price process without making implicit assumptions about how shocks to the expected terminal payoff \tilde{x} correlate with shocks to risk, σ_t .

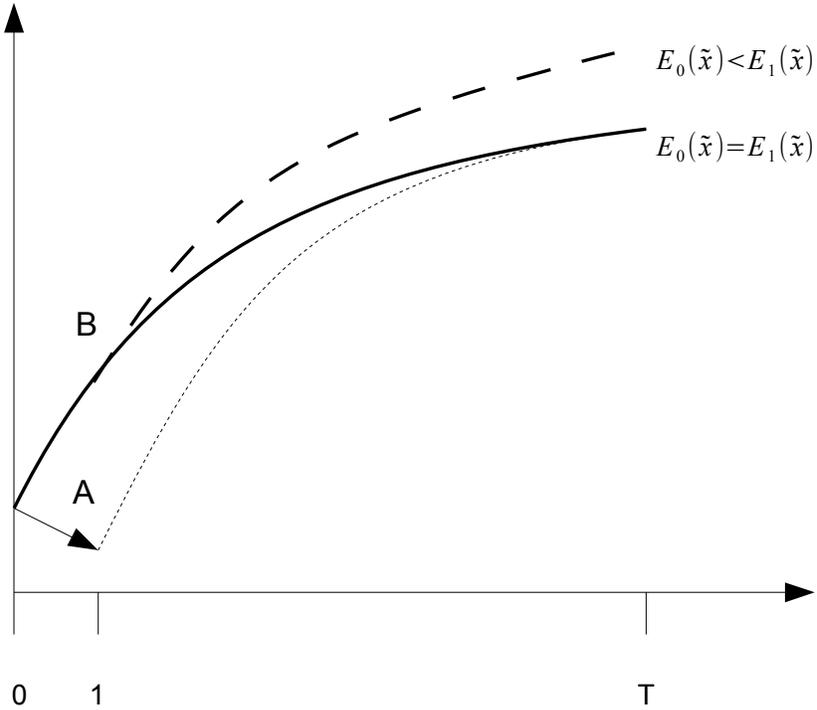


Figure 1: Possible price paths with and without volatility feedback.

The crucial point of our argument is that in an appropriately specified equilibrium price process temporal variations in expected rates of return should be given by changes in the market price of the risky asset. If there is a positive shock to expected returns, the price needs to adjust *negatively* to this shock. We illustrate this in Figure 1. The solid line represents the expected path at time 0. At time 1, there is a positive, unanticipated shock to volatility $\sigma_1^2 - \sigma_0^2 > 0$, giving an increase in the required expected return going forward. There are two ways in which the investor can receive a higher expected return. By keeping the expected terminal payoff unchanged, $E_1(\tilde{x}) = E_0(\tilde{x})$, the price needs to adjust downward to point A in order to reflect the higher expected returns going forward. This adjustment is possible if and only if the price process incorporates an explicit *equilibrium volatility feedback* term in which the price moves by exactly the right amount to generate the steeper expected price path going forward, as illustrated with the dotted path.

The second possibility is that the expected terminal payoff increases. If there is no impact of the increase in volatility on the price, it continues along the path anticipated at time zero (steady state), and thus reaches point B in the figure. Since the expected rate of return between time

1 and T is now higher without the price being adjusted, it must be that the expected terminal value increases, $E_1(\tilde{x}) > E_0(\tilde{x})$. In other words, a positive shock to expected return at time 1 is only consistent with equilibrium if the future expected terminal payoff increases as well.

Our simple figure illustrates that without an explicit equilibrium volatility feedback in the price process, the model will necessarily imply a positive correlation between the shocks to expected returns and the shocks to expected future cash flows. Accordingly, any model that does not explicitly incorporate dynamic volatility feedback effects in the specification of the price process will *at best* be making implicit assumptions about a positive covariation between shocks to expected returns and future cash flows, and in the worst case be inconsistent with equilibrium in the sense that it breaks the present value computations.

We formalize this intuition in the following.

Consider a simple three-period economy. A single risky asset trades at dates 0 and 1. This asset provides the representative investor with an aggregate terminal payoff, \tilde{x} , which she will receive at date 2. Let P_0 and P_1 denote the price of this claim at times 0 and 1. Denote by $R_t = P_{t+1}/P_t$ the gross return. We assume the model to be consistent with present value computation, giving the terminal condition $P_3 = \tilde{x}$. We assume that the expected rates of return are stochastic. At time 1, the investor receives news that may require her to demand a higher or lower expected rate of return going forward. The expected return in the first period, μ_1 is known at time 0. We assume further that the expected return process is in “steady state,” by which we mean that the investor does not anticipate a shock to expected returns at time 1, so $E_0(\mu_2) = \mu_1$.⁴

The investor receives news about the terminal payoff, \tilde{x} and the next period’s expected return μ_2 at time 1. The relative change in the expected return, μ_2/μ_1 and the expected terminal payoff $E_1(\tilde{x})/E_0(\tilde{x})$ are random variables as of time 0. At time 1 both random variables are known. We are interested in the covariation between these random variables and the following theorem characterizes their relationship.

Theorem 1. *Assume that the price process satisfies $\text{Corr}(R_1, \mu_2) = 0$, its expected return process μ_t is random, in steady state $E_0\mu_2 = \mu_1$, and that the price process is consistent with present value computations. Then*

$$\text{Corr}\left(\frac{E_1(\tilde{x})}{E_0(\tilde{x})}, \frac{\mu_2}{\mu_1}\right) > 0. \quad (10)$$

⁴The term steady condition is similar to requiring that the expected return process’ initial condition is its unconditional mean in a multi-period model.

The theorem verifies the simple intuition conveyed in Figure 1. If there is a shock to expected returns at time 1, the future expected cash flow needs to increase to keep the price moving along the steady state path. An immediate corollary follows

Corollary 1. *Under the same assumptions as in Theorem 1, if*

$$\text{Corr}\left(\frac{E_1(\tilde{x})}{E_0(\tilde{x})}, \frac{\mu_2}{\mu_1}\right) = 0, \quad (11)$$

the price process is inconsistent with present value computations.

Consider now how the endogenous price process we derive compares to the assumed price in Merton. We get

$$\frac{dP_t}{P_t} = \lambda_0 dt + \lambda'_1 dX_t + \sigma(X_t) dB_t \quad (12)$$

$$= (\lambda_0^* + \beta' X_t) dt + \lambda'_1 \sigma_x(t) dB_t^x + \sigma(X_t) dB_t \quad (13)$$

for state-variables X_t that follow a stationary, affine process⁵ where β has the usual interpretation as the conditional expected returns' sensitivity with respect to changes in the state-variables. The innovation term dB_t^x captures the shocks to state-variables that may change the expected rates of return, and the innovation term dB_t represents the shocks to the terminal flow. The equilibrium expected returns are thus given by $(\lambda_0^* + \beta' X_t) dt$.

In comparing this equilibrium price to Merton's. He assumes

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma(X_t) dB_t. \quad (14)$$

If the two methods yield the same equilibrium expected return, we have $\mu_t = \lambda_0^* + \beta' X_t$, and

$$\frac{dP_t}{P_t} = (\lambda_0^* + \beta' X_t) dt + \sigma(X_t) dB_t. \quad (15)$$

In comparing (13) to (15) we see that the latter is missing the term

$$\lambda'_1 \sigma(X_t) dB_t^x. \quad (16)$$

This is precisely the term that allows prices to adjust to changes in discount rates/expected rates of return. Without this term, prices are uncorrelated with shocks to expected returns, dB_t^x . In

⁵Specifically, if $dX_t = \mu(X_t)dt + \Sigma(x_t)dB_t^x$ where the drift and diffusion functions can be written $\mu_x(t) = K_0 + K_1 X_t$ and $\Sigma(X_t) = H_0 + \sum_{i=1}^N H_i X_{t,i}$ such that $\beta = \lambda_1 K_1$.

this case, the price process either contains an implicit assumption that future cash flows will increase to offset shocks to discount rates, or the resulting price process in (15) is not consistent with an equilibrium based on present value computations. One may argue that the innovation term dB_t in equation (15) can be taken as a composite effect of dB_t^x and dB_t terms in equation (13). We can write (13) as a scalar Ito process, but the diffusion term of this scalar process depends on the equilibrium parameter λ_1 .⁶ The difference between our model Merton’s in this case is that we also solve for the endogenous diffusion function.

All stochastic present value models will have feedback terms similar to equation (16). For example, in the Bansal-Yaron LRR models the capital gains are given by $\Delta \ln P_t = \Delta \ln D_t + B' \Delta X_t$ where X_t is a vector of state-variables (i.e., expected consumption growth and stochastic volatility in their basic model). Thus, in their model the innovations to the priced risk factors X_t will have immediate, contemporaneous impact on prices. Eraker and Shaliastovich (2008) generalize the BY model to a continuous time-setting with general state-variables and recover equilibrium stock price dynamics with the form of (12). These contemporaneous terms imply that both cash flow news, dB_t , and discount rate news, dB_t^x , impact prices. This gives a “bad-beta/good-beta” interpretation as in Campbell and Vuolteenaho (2004). Our argument implies that properly specified asset pricing models should have shocks that have a “bad-beta/good-beta” interpretation. One implication of our argument is that at least two risk factors are priced in a dynamic model. In the next section we give an example of such a model.

2.4 Conditional CAPM

A large body of empirical articles are concerned with empirical tests of CAPM models with time-varying expected returns. In discrete time, the model is typically referred to as the conditional CAPM. The literature seems to have originated from Merton (1980) who estimated γ based on equation 9. French, Schwert, and Stambaugh (1987), Bollerslev, Engle, and Wooldridge (1988), and Bali and Engle (2010a) also follow Merton (1980) and estimate γ based on equation 9. Tests of the conditional CAPM using additional cross-sectional information include Gibbons and Ferson (1985), Engle, Lilien, and Robins (1987), French, Schwert, and Stambaugh (1987), Nelson (1988), Harvey (1989), Engle, Ng, and Rothschild (1990), Harvey (1991), Ng (1991), Bodurtha and Nelson (1991), Ferson and Harvey (1993), Glosten, Jagannathan, and Runkle (1993), Whitelaw (1994), Evans (1994), Moskowitz (2003), Guo and Whitelaw (2006), and Bali and Engle (2010, 2010), among others.

⁶Specifically, $\lambda_1' \sigma_x(t) dB_t^x + \sigma(X_t) dB_t = \sigma^*(t) dB_t^*$ for $\sigma^*(t) = \sqrt{\text{Tr} \sigma_x(t) \sigma_x(t)' + \text{Tr} \sigma(X_t) \sigma(X_t)'}$ and $B_t^* = (\lambda_1' \sigma_x(t) dB_t^x + \sigma(X_t) dB_t) / (\sqrt{\text{Tr} \sigma_x(t) \sigma_x(t)' + \text{Tr} \sigma(X_t) \sigma(X_t)'})$ (element by element).

The problem with the conditional CAPM literature is that the model itself has no theoretical foundation other than Merton's (1973) analysis. We have argued, as did Hellwig (1977), that Merton's analysis is problematic because it does not verify that the assumed price dynamics are consistent with equilibrium under reasonable assumption about how future cash flows covary with expected returns and assumptions about the supply of risky assets. In the next section we derive an equilibrium model which is based on a subset of the same assumptions as in Merton, but where we relax Merton's exogenous price assumption. As we shall see, this model does not yield Merton's ICAPM unless one considers the special case of constant investment opportunities.

3 The Present Value CAPM

In the following section we discuss our equilibrium dynamic asset pricing framework. We consider first an economy with a single risky asset. This asset does not pay any intermediate dividends. The asset has a terminal value at some fixed date T of \tilde{x}_T . Absence of arbitrage implies $P_T = \tilde{x}_T$. We take P_t to be the market value of \tilde{x}_T at time t . \tilde{x}_T is a random variable, and we will assume that this variable follows some stochastic process. We assume that investors learn about \tilde{x}_T over time. One such example would be a random walk specification for \tilde{x} would imply $E_t \tilde{x}_T = \tilde{x}_t$ such that the current level of \tilde{x}_t is interpretable as the expected future wealth.

We do not think of \tilde{x}_t as a market value or wealth process. Rather, we take \tilde{x}_t to be some exogenous process that converges to terminal wealth. In a single firm or investment project, it is natural to think of the terminal payoff as a cash distribution generated by an irreversible investment. In the aggregate economy we think of \tilde{x} as a portfolio of such irreversible investments where intermediate cash flows are re-invested. The idea is that investors cannot disinvest physical capital invested through corporations in response to increases in discount rates. This setup gives an implicit sluggishness in physical capital, reflected in a completely exogenous terminal cash flow, \tilde{x}_T . It is understood that in general $P_t \neq \tilde{x}_t$ for all $t \neq T$. By *not* forcing the terminal wealth process, \tilde{x}_T , to equal its present value at all times t , we are avoiding any implicit assumptions about the dynamics of its market price. This allows prices to be derived completely endogenously.

The representative investor has utility of terminal wealth,

$$E_t u(\tilde{x}_T). \tag{17}$$

We assume that $E_t u'(\tilde{x}_T) > 0$ and $E_t u''(\tilde{x}_T) < 0$ for all t .

There are no intermediate dividends. The return from holding the market from time t to T is therefore

$$R_{t:T,m} = \tilde{x}_T / P_t$$

where P_t is the market value of the terminal payoff \tilde{x}_T . We have $P_T = \tilde{x}_T$.

Let

$$R_{t:T}(\alpha) = \alpha R_{t:T,m} + (1 - \alpha) R_{t:T,f} \quad (18)$$

denote the gross return to a buy and hold portfolio initiated at time t . The term $R_{t:T,f}$ is the gross risk free rate of return from time t to T . We assume that the risk free return is independent of \tilde{x}_T . If we assume a short rate process, $R_{t,f}$, representing the risk free return over a unit of time, we can write

$$R_{t:T,f} = E_t \prod_{s=t+1}^T R_{s,f}.$$

To facilitate direct comparison with the traditional one period CAPM, it can be useful to assume that the path of interest rates is deterministic, or that the rates are constant. Note that $R_{t:T,f} = E_t \prod_{u=t+1}^s R_{u,f} E_s \prod_{u=s+1}^T R_{u,f} = R_{t:s,f} R_{s:T,f}$.

We define an equilibrium as a price process P_t that clears the market $\alpha_t = 1$ and maximizes investors' utility.

Proposition 2. *The equilibrium market price P_t of the aggregate wealth claim \tilde{x}_T is given by*

$$P_t = \frac{E_t \{u'(\tilde{x}_T) \tilde{x}_T\}}{E_t \{u'(\tilde{x}_T)\} R_{t:T,f}}, \quad \forall t \in [0, T]. \quad (19)$$

The price of an arbitrary asset, $P_{t,i}$, with terminal payoff $\tilde{x}_{T,i}$ in infinitesimal supply is

$$P_{i,t} = \frac{E_t \{u'(\tilde{x}_T) \tilde{x}_{T,i}\}}{E_t \{u'(\tilde{x}_T)\} R_{t:T,f}}, \quad \forall t \in [0, T]. \quad (20)$$

These prices obtain irrespectively of the frequency of trade and portfolio rebalancing. The pricing kernel is

$$\xi_t = \frac{u'(\tilde{x}_T)}{E_t \{u'(\tilde{x}_T)\} R_{t:T,f}}. \quad (21)$$

The Euler equation can also be written in multi-period return form as

$$E_t \{u'(\tilde{x}_T) R_{t:T,m}\} = E_t \{u'(\tilde{x}_T) R_{t:T,i}\}. \quad (22)$$

This Euler equation appears in a slightly different form in Merton and Samuelson (1969) in the context of pricing warrants. It follows directly from the first order conditions for optimal portfolio allocation. It can also be derived by dynamic programming. The fact that this price process obtains irrespectively of the frequency of portfolio rebalancing implies that the Euler equation holds for both continuous and discrete time trading schemes.

A few notes on equations (19) and (20) are in order. These equations contain ratios of expectational expressions. The key to developing tractable pricing models is to be able to compute these expectations. Below we will show that this is easy in the context of the usual workhorse class of affine processes. It is important to see that our model differs from the usual case where P_t equals \tilde{x}_t for all t . We specifically do not impose this restriction. If we did, we would just be reverting to Merton's solution because our price dynamics would be entirely driven by the assumed dynamics for \tilde{x}_T .

3.1 Power Utility

Our theory will give closed form expressions for the evolution of stock prices in economies where the representative agent has power utility of wealth,

$$u'(\tilde{x}_T) = \tilde{x}_T^{-\gamma}$$

and the process that generates the final payoff, \tilde{x}_T , is driven by exponential affine processes. This means that we can write

$$E_t u'(\tilde{x}_T) = E_t \tilde{x}_T^{-\gamma} = \exp(-\gamma \ln \tilde{x}_T) = \exp(\alpha(\gamma, t, T) + \beta(\gamma, t, T)' X_t) \quad (23)$$

for some exogenously specified, stationary process X_t . where α and β solve Riccati ODE's (see appendix B).

Consider a representative agent with power utility and suppose that the conditional moment generating function of $\ln \tilde{x}_T$ is known and given by equation (23). Equation 19 now gives the general expression

$$\begin{aligned} \ln P_t &= \alpha(1 - \gamma, t, T) - \alpha(\gamma, t, T) - [\beta(1 - \gamma, t, T) - \beta(-\gamma, t, T)]' X_t - R_{t,T,f} \\ &=: \lambda_0(\gamma, t, T) + \lambda_x(\gamma, t, T)' X_t - r_{t:T,f} \end{aligned} \quad (24)$$

where $r_{t:T,f} = \ln R_{t:T,f}$ is the continuously compounding risk free return from t to T . We can think of λ_x as representing factor sensitivities. Unlike multi-factor no-arbitrage models as the class considered by Ross (1976), factor loadings, λ_x , in our model, are not free parameters. They depend on preferences and importantly, on the dynamic characteristics of the state-variables X . We demonstrate this in the example below.

3.2 An Example Stochastic Volatility Model for the Market Portfolio

To illustrate our approach, consider the following model for the terminal exogenous wealth \tilde{x}_T

$$\frac{d\tilde{x}_t}{\tilde{x}_t} = \mu dt + \sigma_t dB_t, \quad (25)$$

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \sigma_v \sigma_t dB_t^v + \xi_t dN_t \quad (26)$$

where we assume the standard affine state dependent jump arrival intensity, $l(\sigma_t^2) = l_0 + l_1 \sigma_t^2$ where l_0, l_1 are parameters. The jump sizes are assumed to be exponentially distributed, as in Duffie, Pan, and Singleton (2000).

The state-variable X_t in this economy is defined $X_t = (\ln \tilde{x}_t, \sigma_t^2)$. The equilibrium price process is now characterized by a solution to the coefficients in equation (24). The model's equilibrium coefficients solve a system of Riccati ordinary differential equations. We shall discuss some special cases. In particular, we first consider the case where the variance process σ_t^2 is a pure jump process with constant arrival intensity. In the appendix we show that the equilibrium coefficients are available in closed form as

$$\begin{aligned} \lambda_\sigma(\gamma, t, T) &= \beta_\sigma(1 - \gamma, t, T) - \beta_\sigma(-\gamma, t, T) \\ &= (e^{-\kappa(T-t)} - 1) \frac{\gamma}{\kappa}. \end{aligned} \quad (27)$$

Figure 2 shows the value of λ_σ for different investment horizons. The figure is computed using a volatility speed-of-mean-reversion $\kappa = 0.01$, which is typical of estimates found in daily aggregate financial market time series. As can be seen from the figure, the volatility sensitivity is essentially constant up to a few years prior to the terminal payoff. Thus, unless the investor is close to the termination date, the volatility sensitivity is given by $-\gamma/\kappa$, which is the infinite horizon limit of the model. We discuss the infinite horizon limit in more detail below.

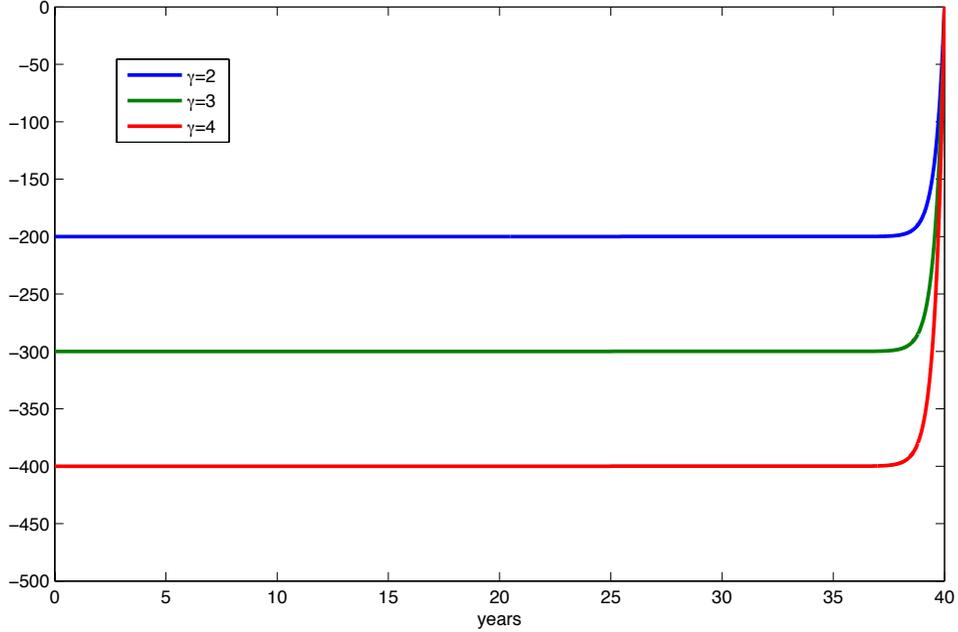


Figure 2: The volatility sensitivity, λ_σ , for a $T = 40$ year investment horizon.

3.3 Infinite Horizon Limit

It is useful to consider the infinite horizon limit of our model for two reasons. First, in the infinite horizon limit return processes are stationary. Second, it is easy to find the equilibrium coefficients, λ_x . To see this, we note that any economy in which the fundamental $\ln \tilde{x}_T$ depends on a set of stationary state-variables we have

Proposition 3. *If the infinite horizon limit exists, the value of the market portfolio is given by (24) where the coefficients α and β solve the non-linear equation system*

$$0 = \mathcal{K}'\beta + \frac{1}{2}\beta'H\beta + l_1'(\varrho(\hat{\beta}) - 1), \quad (28)$$

$$\alpha(s) = \left(\mathcal{M}'\beta + \frac{1}{2}\beta'h\beta + l_0'(\varrho(\beta) - 1) \right) s \quad (29)$$

where $\beta = \beta(u, \infty)$ is a constant vector.

Thus, in the infinite horizon limit the usual Riccati equations associated with affine asset pricing models reduce to a quadratic, matrix-valued equation. It is straightforward to verify that

eqn. (28) reproduces the solution $\beta_\sigma(\infty) = \gamma/\kappa$. Using (29) we find that the example model in (25) and (26) gives the constant term

$$\lambda_0 = (\theta\gamma + l_0 ((1 - \mu_\xi\beta_\sigma(1 - \gamma))^{-1} - (1 - \mu_\xi\beta_\sigma(-\gamma))^{-1})). \quad (30)$$

This term is interpretable as a steady-state, unconditional risk premium, $E(d \ln P)/dt - rf$.

We can now apply Ito's lemma to (24) to find

$$d \ln P = \lambda_0 dt - \frac{\gamma}{\kappa} d\sigma_t^2 + \sigma_t dB_t \quad (31)$$

Equation (31) is the continuous time analogue of equation (1) from the introductory section. We now examine this equation in more detail.

Figure 3 plots a simulated sample path for the exogenous terminal payoff \tilde{x}_t , its volatility σ_t , and the endogenous returns and price relative to the fundamental, \tilde{x}_t . We use a finite horizon economy on purpose, illustrating that the price process converges to the fundamental \tilde{x} at the final date, T . Remember that $P_t = \tilde{x}_t$ if there is no risk premium. An increase in risk leads to an increase in the risk premium, and hence lowers the P/\tilde{x} ratio. The particular sample path shown in the figure has a clustering of volatility jumps in the middle time period. These volatility shocks are barely visible in the "fundamental" \tilde{x}_t shown on the bottom right, but very clearly affect realized stock returns, shown on the bottom left. The volatility jumps are clearly visible in the endogenous returns and lead to large negative returns (less than negative twenty percent), mimicking crashes. The upper right plot shows the ratio of the price P_t to the fundamental \tilde{x}_t . This ratio is interpretable as a discount. The price P is below \tilde{x} if and only if investors are risk-averse. The P/\tilde{x} ratio is plotted for different values of risk aversion γ , and shows clearly how larger risk aversion leads to larger stock market crashes in response to volatility jumps. Higher values of γ also give a higher unconditional return, as illustrated by the lower initial price (relative to \tilde{x}_0). The unconditional expected return is given by λ_0 which is generally an increasing function of γ as well as the parameters that govern the volatility dynamics. The expected rate of return is higher than the unconditional average, λ_0 , whenever σ_t^2 exceeds its long run mean and vice versa.

It is also worthwhile to notice that in this simple model, σ_t^2 is not the variance of the market portfolio. Rather, taking the conditional variance of both sides of (31) we get $\text{Var}(d \ln P)/dt = A + B\sigma_t^2$ in general. This means that the expected rate of (logarithmic) return for the aggregate stock market can be written as a linear function of the stock market variance, but does not obey a direct proportionality relationship such as in (9). This has implications for empirical inference,

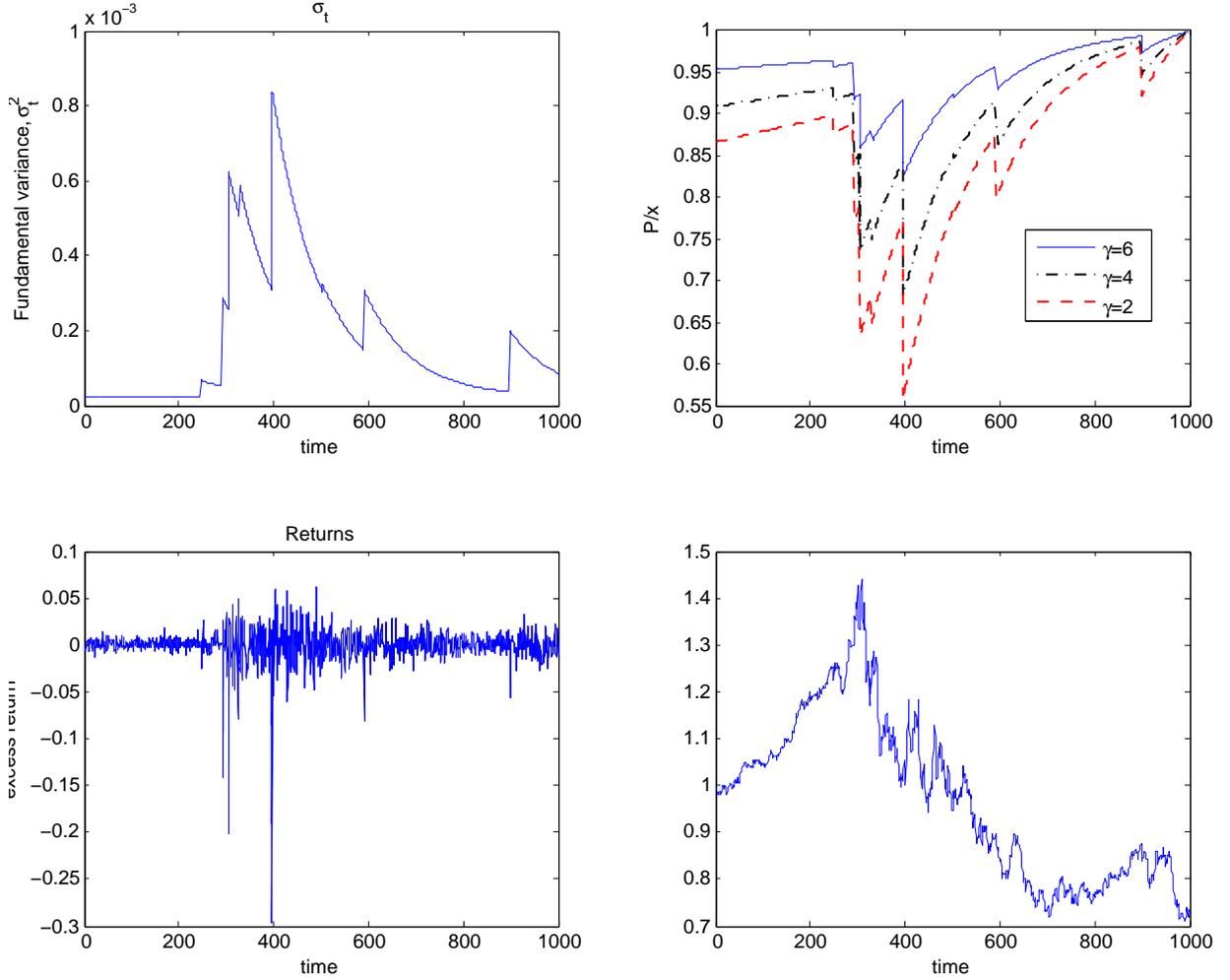


Figure 3: Asset pricing implications for the aggregate market model

as the econometrician needs to obtain some estimate of σ_t^2 rather than $\text{Var}_t(\ln R_t)$ in order to estimate γ .

Another special case of the stochastic volatility process (26) is obtained by setting the jump term to zero. In this case the volatility process follows a square root process as in Heston (1993). The equilibrium stock price process is given by

$$\frac{dP_t}{P_t} = \left(\frac{1}{2}\lambda_m^2\sigma_v^2 - \lambda_m\kappa\right)\sigma_t^2 dt + \lambda_m\sigma_v\sigma_t dB^\sigma + \sigma_t dB \quad (32)$$

where

$$\lambda_m = \sqrt{\kappa^2 - \sigma_v^2(\gamma^2 + \gamma)} - \sqrt{\kappa^2 - \sigma_v^2(\gamma^2 - \gamma)} \quad (33)$$

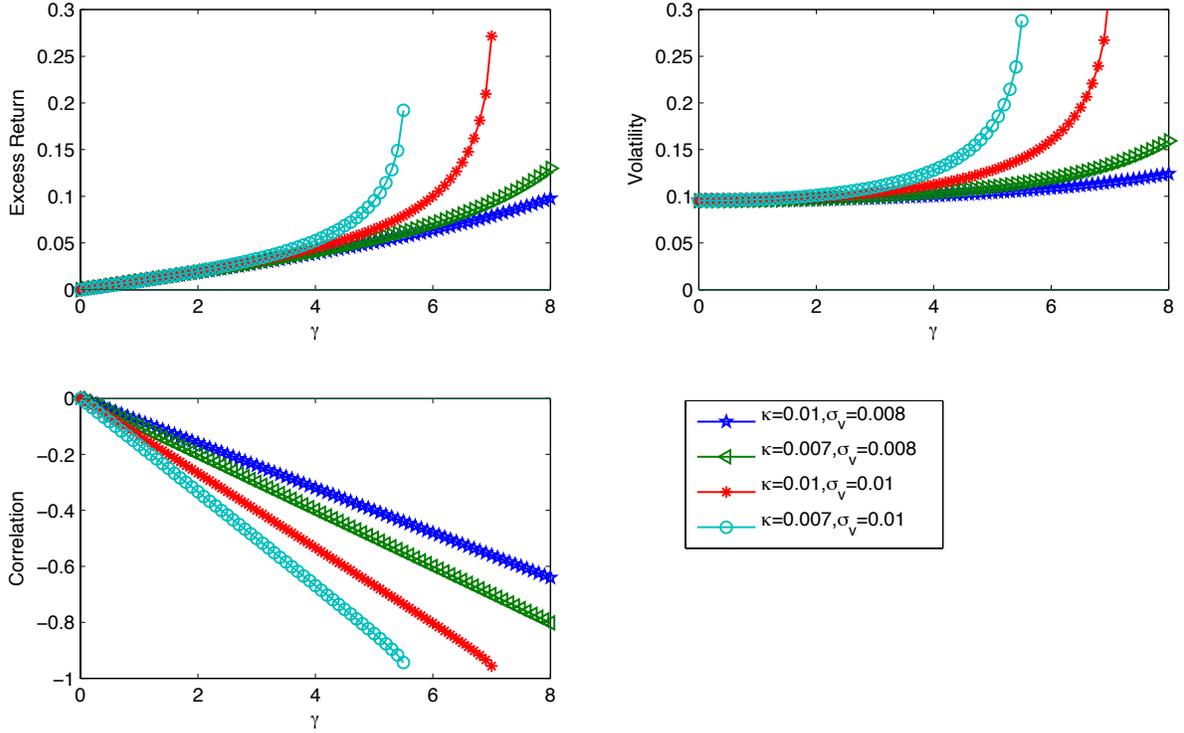


Figure 4: The figure shows annualized expected excess returns, volatility, and the correlation between changes in volatility and returns as a function of risk aversion, γ . The plots are based on empirically reasonable values for the speed of mean reversion, κ , and volatility-of-volatility, σ_v .

is the volatility sensitivity of the stock price.

The strong negative relationship between market returns and changes in volatility suggests that we can exploit this contemporaneous relationship to recover more accurate empirical estimates of γ than what is typically found by regressing future returns onto lagged conditional market variance. Figure 4 shows how unconditional moments in our model depend on risk aversion, γ , for different values of the model parameters θ, κ and σ_v . As can be seen from the bottom plot, it is evident that the model is capable of producing a very large negative correlation between innovations in volatility and returns. As this correlation is typically estimated to be in the range -0.5 to -0.9 range, we see that γ 's exceeding 4 are consistent with these estimates. The figure also illustrates that the model can generate a very substantial equity premium, and some even exceed the empirically plausible values. The model also produces empirically plausible annualized volatility numbers that generally range from 10% to 20%. An example parameter constellation that produces empirically plausible values are $\theta = 0.0075^2, \kappa = 0.007, \sigma_v = 0.0008$

and $\gamma = 5$. These parameters imply an expected (log) excess rate of return of 9%, volatility of 14.5% and a correlation of -0.57 . A full empirical analysis of this model will be reported in future work.

3.4 Comparison to Existing Models

3.4.1 Merton's ICAPM

Our example model coincides with Merton's ICAPM as a special case only when σ_t is constant. When σ_t is random, our model is different in two important practical ways:

- There are at least two priced risk factors.
- The market portfolio is not a priced risk factor.

In our example model, shocks to expected future cash flows, dB_t , are priced independently from shocks to the stochastic volatility process. If we include additional random factors such stochastic interest rates, shocks that drive these processes will also be priced giving at least two priced risk factors.⁷ The return on the market portfolio is not a priced factor in our model because the market return itself is driven by two priced risk factors. The standard market beta, conditionally or unconditionally, is not a sufficient statistic in characterizing returns to risky assets.

It is important to emphasize that our modeling assumptions are special cases of Merton's, with the exception that we do not make any assumptions about the structure of the price process. Yet our equilibrium price process is very different from those assumed in Merton. Moreover, if the economy is driven by pure diffusion processes, our equilibrium price endogenizes both the expected instantaneous return as well as the diffusion process that governs the equilibrium price process.

3.4.2 Long Run Risk

Our model is connected to long run risk in important ways. In particular, our return process in eqn. Equation (31) contains the volatility feedback term where $-\gamma/\kappa$ is the volatility factor loading. This volatility feedback term decreases in risk aversion and explodes as the speed of

⁷Jagannathan and Wang (1996) point out that a single factor discrete time ICAPM can be interpreted as having two factor unconditional CAPM factor structure. By the same argument our model has at least two priced risk factors conditionally and may generate as many or more factors unconditionally as conditionally.

mean-reversion κ becomes small. κ can be approximated as one minus the first order autocorrelation of volatility so the volatility factor loading explodes as volatility autocorrelation approaches unity.

The fact that the volatility factor loading increases with the reciprocal of κ is reminiscent of the equilibrium factor loadings produced by LRR models. In fact, both models produce a volatility feedback term which is proportional to $-\gamma/\kappa$.⁸ Thus, our model reproduces the key aspect of long-run-risk models - the longer the impact of a shock, the higher the associated equilibrium risk premium. The long run risk effects will also show up if we modify our model to have additional persistent state-variables. For example, including an AR(1)-like process (i.e. OU process) for the conditional mean of \tilde{x}_t will give a pricing effect similar to expected consumption in Bansal & Yaron.

How is it that our model produces these long run risk effects? To answer this, it is important to understand why those effects show up in the BY model in the first place. In LRR, the aggregate wealth is defined as the present value of future aggregate consumption. The key is to see that our utility-of-terminal wealth framework can be written as a special case of Epstein-Zin utility when the agent is indifferent to the timing of his or her consumption flows. We now formalize this insight.

As above, we assume that the economy is driven by a terminal payoff \tilde{x}_T and state-variables X_t . Epstein-Zin utility is defined through the recursion

$$U(\tilde{x}_t, X_t) = \{(1 - \beta)u_t^{1-\frac{1}{\psi}} + \beta[E_t(U(\tilde{x}_{t+1}, X_{t+1})^{1-\gamma})]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\}^{\frac{1}{1-\frac{1}{\psi}}}$$

where the per-period utility u_t typically depends on intermediate consumption, say c_t . The parameters β , γ and ψ represent the subjective discount factor, risk aversion, and IES, respectively. Without intertemporal consumption, we have $u_t = 0$ and the following result holds:

Proposition 4. *Our terminal utility of wealth model*

$$V_t = \delta^{T-t} E_t \left\{ \frac{\tilde{x}_T^{1-\gamma}}{1-\gamma} \right\}$$

is a special case of EZ when $\lim \psi \rightarrow \infty$ and $u_t = 0$ (no utility from intermediate consumption).

⁸In BY's expression for A_2 on p. 1487, remove the expected consumption growth, x_t , by setting $\rho = \phi_e = 0$, and take the limit $\psi \rightarrow \infty$ in their model. The approximate identity $\kappa = 1 - \nu_1$ where ν_1 is first order volatility autocorrelation in the BY model gives $A_2 = \frac{1}{2}(1 - \gamma)/\kappa$.

Since the utility of terminal wealth specification is a special case of Epstein-Zin, we expect that present value computations based on terminal utility of wealth look similar to present values computed in an Epstein-Zin utility framework with large IES. This is of course precisely what we observe in our model.

Our model still has a large advantage over present value models based on discrete time dividend discounting. It does not require linearization of the return process as in Campbell and Shiller (1987). The Campbell-Shiller linearizations lead to approximation errors. The linearization coefficients can also be hard to compute numerically which again leads to difficulties in empirical estimation.⁹ Therefore our model is considerably easier to implement than LRR models.

3.4.3 Reduced Form Models

Consider adding a quick fix to the problems described by allowing for a pre-specified correlation term between the volatility shocks and the returns. For example, Heston (1993) proposes a stochastic volatility model where the shocks to the volatility process are correlated with price shocks. The correlation is assumed to be constant.

This solution is a reduced form approach which will yield a price process that is consistent with equilibrium only in a few cases. Consider the case when our example volatility process contains both jump and diffusive components, where the former has intensity that is a linear function of the volatility level (i.e., arrival intensity parameters l_0 and l_1 and the volatility-of-volatility parameter σ_v are assumed to be strictly positive.) In this case it is easy to show that the equilibrium correlation between volatility and price shocks is a non-linear function of σ_t in our model. This is inconsistent with the constant correlation in Heston's (1993) model.

Even if we specify a reduced-form model that happens to be consistent with equilibrium, it is not obvious what a particular numerical choice for ad hoc correlation terms should be. Thus, in general, it is unlikely that a reduced form guess at a model structure will yield a model solution or estimated model parameters that are consistent with equilibrium.

⁹The linearization coefficients can be computed through a numerical solution to a fixed point problem. The computation of the linearization coefficients should be done for each parameter update in a non-linear structural estimation of LRR models, as in Bansal, Kiku, and Yaron (2006).

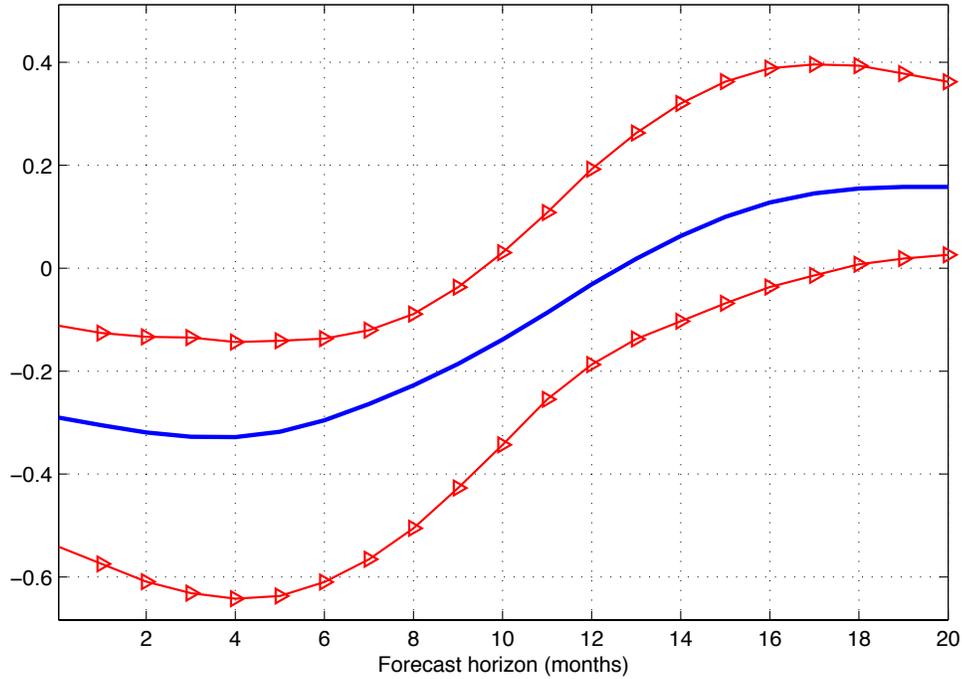


Figure 5: Conditional stock market volatility and earnings growth: The figure shows the predictive ability of stock market volatility for subsequent aggregate earnings growth rates. 95% pointwise confidence intervals were computed using 60 month block bootstraps.

4 Conditional Beta

How does financial market volatility impact the real economy? It is quite reasonable to postulate an equilibrium in which firms face higher financing costs when markets are volatile. Both equity and debt financing become more costly when volatility is higher. This can have real effects on firms' cash flows. Also, economic uncertainty could curb spending, leading to a negative aggregate demand shock. Campbell, Lettau, Malkiel, and Xu (2001) and Fornari and Mele (2011) present empirical evidence suggesting that financial market volatility predicts a substantial fraction of the variation in aggregate output growth. Figure 5 shows the correlation between conditional volatility and subsequent earnings growth in the data. The figure illustrates that earnings growth rates are significantly negatively correlated with past volatility. The correlation is strongest at about a four month forecasting horizon.

A full structural model of how financial market risk impacts firm financing, and thus earnings growth, is beyond the scope of this paper. We consider the following “reduced form” model

$$\begin{aligned}\frac{d\tilde{x}_t}{\tilde{x}_t} &= (\mu_m + \phi_m \sigma_{m,t}^2)dt + \sigma_{m,t}dB_{m,t} \\ d\sigma_{m,t}^2 &= \kappa_m(\theta_m - \sigma_{m,t}^2)dt + \sigma_{v,m}\sigma_{m,t}dZ_{m,t}\end{aligned}$$

The cash flow process and the idiosyncratic stochastic volatility process for asset i are specified as

$$\begin{aligned}\frac{d\tilde{x}_{i,t}}{\tilde{x}_{i,t}} &= (\mu_i + \phi_i \sigma_{m,t}^2)dt + \sigma_{i,t}dB_{i,t} + C_i \sigma_{m,t}dB_{m,t} \\ d\sigma_{i,t}^2 &= \kappa_i(\theta_i - \sigma_{i,t}^2)dt + \sigma_{v,i}\sigma_{i,t}dZ_{i,t}\end{aligned}$$

The coefficients, ϕ and ϕ_i measure the sensitivity of fundamentals to the systematic volatility, $\sigma_{m,t}$. If these coefficients are negative, shocks to volatility lead to subsequent lower growth rates. If $\phi_i < \phi$, the cash flow sensitivity of stock i is greater than that of the average stock in the economy. Individual assets also have asset-specific sensitivity to shocks in volatility as captured by the term $C_i \sigma_{m,t}dB_{m,t}$. Thus, we can think of the ϕ_i coefficients as capturing the sensitivity of long run expected cash flows to systematic volatility, and C_i as capturing the sensitivity of firm specific cash flows to aggregate cash flows, much like a beta in a standard CAPM model.

Under these assumptions about exogenous fundamentals, we show in Appendix A that the equilibrium price processes given by

$$\frac{dP_{m,t}}{P_{m,t}} - r_{f,t}dt = \left(\frac{1}{2}\lambda_m^2 \sigma_{v,m}^2 - \lambda_m \kappa_m\right)\sigma_{m,t}^2 dt + \sigma_{m,t}dB_{m,t} + \lambda_m \sigma_{v,m}\sigma_{m,t}dZ_{m,t} \quad (34)$$

and

$$\frac{dP_{i,t}}{P_{i,t}} - r_{f,t}dt = \left(\frac{1}{2}\lambda_i^2 \sigma_{v,m}^2 - \lambda_i \kappa_m\right)\sigma_{m,t}^2 dt + \sigma_{i,t}dB_{i,t} + C_i \sigma_{m,t}dB_{m,t} + \lambda_i \sigma_{v,m}\sigma_{m,t}dZ_{m,t} \quad (35)$$

where the coefficients, λ_m and λ_i are given by

$$\lambda_m = \frac{1}{\sigma_{v,m}^2} \left[\sqrt{\kappa_m^2 - \sigma_{v,m}^2(\gamma^2 + (1 - 2\phi_m)\gamma)} - \sqrt{\kappa_m^2 - \sigma_{v,m}^2[\gamma^2 - (1 + 2\phi_m)\gamma + 2\phi_m]} \right] \quad (36)$$

and

$$\lambda_i = \frac{1}{\sigma_{v,m}^2} \left[\sqrt{\kappa_m^2 - \sigma_{v,m}^2(\gamma^2 + (1 - 2\phi_m)\gamma)} - \sqrt{\kappa_m^2 - \sigma_{v,m}^2[\gamma^2 + (1 - 2C_i - 2\phi_m)\gamma + 2\phi_i]} \right] \quad (37)$$

The traditional conditional β_i , defined as the ratio of the conditional covariance of individual asset returns with the market divided by market variance, is equal to

$$\beta_i = \frac{C_i + \lambda_i \lambda_m \sigma_{v,m}^2}{1 + \lambda_m^2 \sigma_{v,m}^2}.$$

The conditional betas are constant with this particular model specification. More generally, the betas will be ratios of linear functions of state-variables. It is generally not true that risk premiums are linear functions of market beta in our model. Figure 6 illustrates this. This figure plots expected returns as functions of the coefficients C_i and ϕ_i under two different assumptions about the initial value of the market volatility $\sigma_{m,t}$. The figure shows that the expected rates of return look as if they are almost linear functions of ϕ 's and C 's. Also, the expected returns are differ greatly across the two different initial volatility regimes, suggesting that expected rates of return can vary greatly over time in our model. In periods of low volatility, there may be a discernible risk premium and therefore it may be difficult to detect a strong empirical relationship between average returns and systematic risk during these periods. Conversely, in periods of high volatility we should see a greater overall market risk premium, as well as a greater cross-sectional dispersion in expected returns. When volatility moves up, high-risk stocks lose value relative to their low risk counterparts, and vice versa. These implications are potentially important for cross-sectional tests of asset pricing models. Ang, Hodrick, Xing, and Zhang (2006) find that volatility predicts cross-sectional return premiums.

5 Concluding Remarks

Merton's ICAPM is based on the idea that by postulating an exogenously given geometric process, one can solve for the period-by-period expected rate of return in order to derive the equilibrium price process. The problem with this approach is that the expected rate of return serves two purposes. It is the average return per period but also the discount rate. The practice in academic finance of simply solving for the expected rate of return ignores the issue of *where expected returns come from*. As soon as one fixes the cash flow paid by an asset, whether it is a perpetual dividend stream as with stocks, a state-contingent claim as with an option, or a fixed income stream as in a bond, the present value *should* depend negatively on discount rates. This holds for all arbitrage free bond pricing models and stock pricing models based explicitly on dividend discounting. Any stock price process that claims to represent an equilibrium with time-varying expected returns that does not incorporate contemporaneous discount rate shocks is either not an equilibrium or is at best an equilibrium under the implicit assumption that future cash flows covary positively

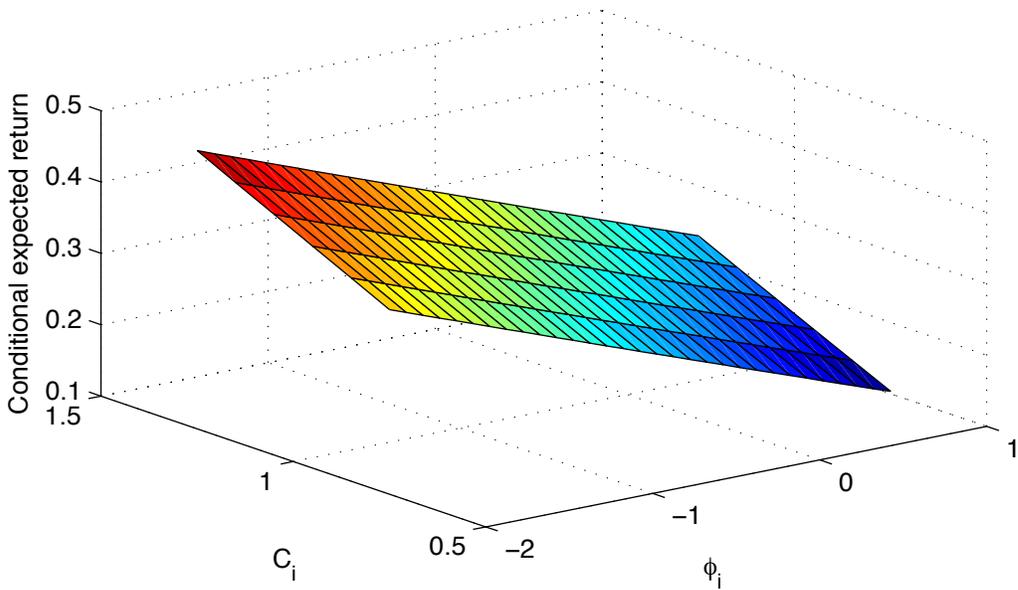
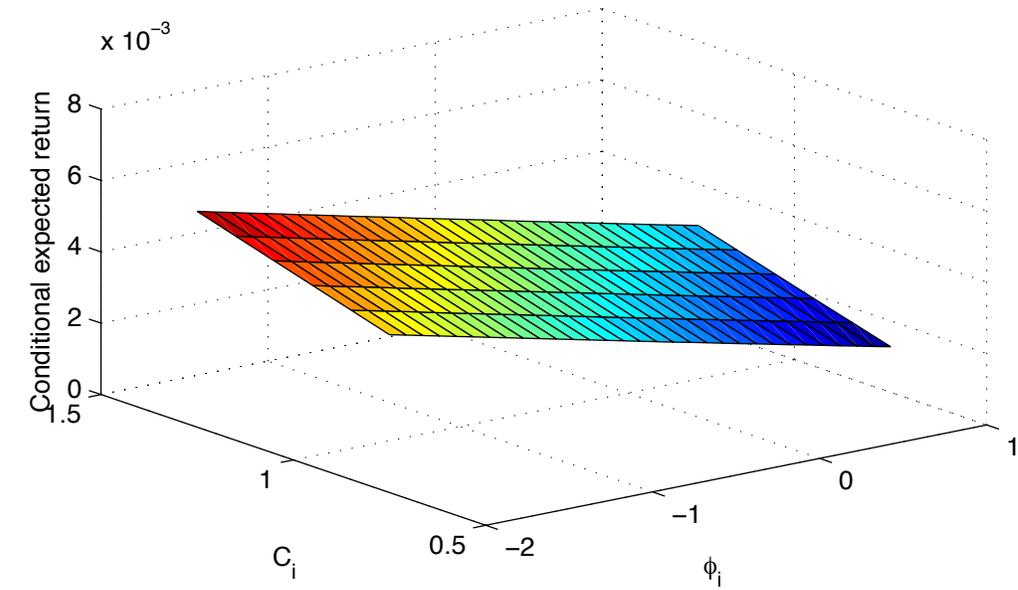


Figure 6: Expected annual rates of return as functions of the factor exposures, C_i and ϕ_i . The upper (lower) graphs represent low (high) initial market volatility, $\sigma_{m,t}$ respectively.

with changes in discount factors, such that any change in the discount factor is exactly offset by an increase in future expected payoffs.

We are not the first to criticize Merton's model. In an unpublished paper Hellwig (1977), shows that Merton's exogenous price assumption is difficult to reconcile with particular choices of dividend and utility specifications. Bick (1990) conducts a theoretical analysis searching for particular utility functions consistent with an assumed equilibrium price process within Merton's ICAPM. The objective of this exercise seems by itself to contradict the assertion that "... the assumptions of continuous trading and diffusion-type stochastic processes justify the use of variance as a sufficient statistic for risk without the objectionable assumptions of either quadratic utility or normally-distributed stock returns" (Merton, 1980, p. 329).

Our critique of Merton's analysis is relevant in the context of empirical implementations of the intertemporal, or conditional CAPM. The direct consequence of our critique is that return processes which have been assumed to be consistent with equilibrium are missing contemporaneous feedback effects, and potentially also ignore priced risk factors associated with changing discount rates. All in all, this could imply that empirical tests of conditional CAPM models are potentially misleading. This is, in some sense, good news, since the empirical evidence on the conditional CAPM is conflicting and not very supportive of the model.

We outline a general multi-period Euler equation that is aimed at developing a dynamic CAPM consistent with present value computations. In doing so we start with a strictly exogenous process that provides some terminal payoff at a terminal date T . By solving the agents' portfolio choice problem we obtain a simple Euler equation which can be used to find the equilibrium price process. The usual combination of exponential affine processes, coupled with power utility of wealth, accordingly produce closed-form equilibrium prices.

Our general idea can be expanded upon in numerous ways. In this paper we have been wanting to keep the modeling as close to the traditional ICAPM as possible, and in doing so, we have not considered the impact of stochastic interest rates. In examining equations (19) and (20) it is clear that under the usual affine process - power utility combination, log returns will have a term containing $-d \ln R_{t:T,f}$ in continuous time. This term thus provides a feedback term to stock prices suggesting that stocks should respond negatively to increases in interest rates. It is possible to generalize this again, for example, by introducing stocks with differing duration characteristics. It is easy to do this by allowing for the terminal payoff date to differ across assets. This introduces different interest rate sensitivities across classes of stocks that have different dividend durations.

It is possible to generalize our model in various other ways. In our basic setup we have assumed that the terminal payoffs \tilde{x}_T and $\tilde{x}_{T,i}$ are exogenous. In our model, these quantities are interpreted as “fundamentals” to which the stock price will eventually converge. We have explicitly assumed that the processes driving these fundamentals are strictly exogenous. In particular, we did not assume that the mean expected growth rate of these processes were somehow determined in equilibrium. This is what led us to gettin a different solution other than the standard ICAPM style models without feedback terms.

It is perhaps difficult to interpret exactly how these terminal payoffs relate to tangible macro economic quantities. As an investment project, $\tilde{x}_{T,i}$ should be interpreted as the date T value of (reinvested) cash flows generated by the project up until that time. It is natural in this case to interpret the investment project as *irreversible*. Our framework should in general be thought of as one where there is some implicit friction on the conversion of physical capital into consumer goods. If not, we could have interpreted $\tilde{x}_{t,i}$ as the value of a firm, in which case $P_{t,i} \neq \tilde{x}_t$ would imply arbitrage. We believe our assumption is reasonable. Corporations do not sell off assets instantaneously to compensate investors for shocks to uncertainty. They respond sluggishly, if at all. Financial prices, on the other hand, respond immediately. The volatility feedback effect implicitly evinces the existence of corporations’ sluggish responses to changes in financial market volatility.

In future extensions of our model it is natural to consider macro-economic models in which cash flows, $\tilde{x}_{t,i}$ are also endogenous. It is natural to think that the cash flows should depend on financing costs, and as such, need to be solved in a bigger general equilibrium setup than we consider here.

References

- Ang, Andrew, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang, 2006, The Cross-Section of Volatility and Expected Returns, *Journal of Finance* 61, 259299.
- Bali, Turan, and Robert F. Engle, 2010a, Investigating ICAPM with Dynamic Conditional Correlations, *Journal of Monetary Economics* 57(4), 377–390.
- Bali, Turan, and Robert F. Engle, 2010b, Resurrecting the Conditional CAPM with Dynamic Conditional Correlations, *Working paper, Georgetown*.
- Bansal, Ravi, Dana Kiku, and Amir Yaron, 2006, Risks for the Long Run: Estimation and Inference, *working paper, Duke*.

- Bansal, Ravi, and Amir Yaron, 2004, Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles, *Journal of Finance* 59, 1481–1509.
- Bick, Avi, 1990, On Viable Diffusion Price Processes of the Market Portfolio, *Journal of Finance* 45, 673–689.
- Bodurtha, James N., and Mark C. Nelson, 1991, Testing the CAPM with time-varying risks and returns, *Journal of Finance* XLVL, 1485–1505.
- Bollerslev, Tim, Robert F. Engle, and Jeffrey M. Wooldridge, 1988, A Capital Asset Pricing Model with Time-Varying Covariances, *Journal of Political Economy* 96, 116–131.
- Campbell, John, 1993, Intertemporal Asset Pricing without Consumption Data, *American Economic Review* 83, 487–512.
- Campbell, John, and Toumo Vuolteenaho, 2004, Bad Beta, Good Beta, *American Economic Review* 94(5), 1249–1275.
- Campbell, John Y., Martin Lettau, Burton G. Malkiel, and Yexiao Xu, 2001, Have Individual Stocks Become More Volatile? An Empirical Exploration of Idiosyncratic Risk, *Journal of Finance* 56, 1–43.
- Campbell, John Y., and Robert J. Shiller, 1987, Cointegration and Tests of Present Value Models, *Journal of Political Economy* 95, 1062–1088.
- Campbell, John Y., and Robert J. Shiller, 1988, The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors, *Review of Financial Studies* 1(3), 195–228.
- Cochrane, John, 2011, Presidential Address: Discount Rates, *Journal of Finance* 66(4), 1047–1108.
- Duffie, D., and R. Kan, 1996, A yield-factor Model of Interest Rates, *Mathematical Finance* 6, 379–406.
- Duffie, Darrell, Jun Pan, and Kenneth J. Singleton, 2000, Transform Analysis and Asset Pricing for Affine Jump-Diffusions, *Econometrica* 68, 1343–1376.
- Engle, R. F., David M. Lilien, and Russel P. Robins, 1987, Estimating the time varying risk premia in the term structure: The ARCH-M model, *Econometrica* 55, 391–407.
- Engle, Robert. F, Victor. K Ng, and Michael Rothschild, 1990, Asset pricing with a FACTOR ARCH covariance structure: Empirical estimates for treasury bills, *Journal of Econometrics* 45, 213–237.

- Eraker, Bjørn, 2010, The Volatility Premium, *working paper*.
- Eraker, Bjørn, and Ivan Shaliastovich, 2008, An Equilibrium Guide to Designing Affine Pricing Models, *Mathematical Finance* 18-4, 519–543.
- Evans, Martin D., 1994, Expected returns, time-varying risk, and risk premia, *Journal of Finance* XLIX, 655–679.
- Ferson, Wayne E, and C. A. Harvey, 1993, The risk and predictability of international equity returns, *Review of Financial Studies* 6, 527–566.
- Fornari, Fabio, and Antonio Mele, 2011, Financial Volatility and Economic Activity, *working paper*, *LSE*.
- French, Kenneth R., William Schwert, and Robert F. Stambaugh, 1987, Expected stock returns and volatility, *Journal of Financial Economics* 19(1), 3–29.
- Gibbons, Michael, and Wayne Ferson, 1985, Testing asset pricing models with changing expectations and an unobservable market portfolio, *Journal of Financial Economics* 14, 217–236.
- Glosten, L. R., R Jagannathan, and D. E. Runkle, 1993, On the relation between the expected value and the volatility of the nominal excess return on stocks, *Journal of Finance* 5, 1779–1801.
- Gordon, Myron J., 1959, Dividends, Earnings and Stock Prices, *Review of Economics and Statistics* 41 (2), 99–105.
- Guo, Hui, and Robert F. Whitelaw, 2006, Uncovering the Risk-Return Relation in the Stock Market, *Journal of Finance* 61, 1433–1463.
- Harvey, Campbell R., 1989, Time-varying conditional covariances in tests of asset pricing models, *Journal of Financial Economics* 24, 289–317.
- Harvey, Campbell R., 1991, The world price of covariance risk, *Journal of Finance* 46, 111–157.
- Hellwig, Martin F., 1977, On the Validity of the Intertemporal Capital Asset Pricing Model, *Center for Operations Research and Econometrics (CORE) Discussion paper* 7744.
- Heston, Steve, 1993, Closed-Form Solution of Options with Stochastic Volatility with Application to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.
- Jagannathan, Ravi, and Zhenyu Wang, 1996, The Conditional CAPM and the Cross-Section of Expected Returns, *Journal of Finance* 51(1), 3–53.

- Merton, Robert, 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867–887.
- Merton, Robert C., 1980, On Estimating the Expected Return on the Market, *Journal of Financial Economics* 8, 323–361.
- Merton, Robert C., and Paul Samuelson, 1969, A Complete Model of Warrant Pricing that Maximizes Utility, *Sloan Management Review* Winter, 17–46. Reprinted in Merton, R. C. “Continuous Time Finance,” Wiley–Blackwell, 1998.
- Moskowitz, Tobias J., 2003, An Analysis of Covariance Risk and Pricing Anomalies, *Review of Financial Studies* 16, 417–457.
- Nelson, Mark, 1988, Time varying betas and risk premia in the pricing of forward foreign exchange contracts, *Journal of Financial Economics* 22, 335–354.
- Ng, Lilian, 1991, Tests of the CAPM with time-varying covariances: A multivariate GARCH approach, *Journal of Finance* XLVL, 1507–1521.
- Ross, Stephen, 1976, The arbitrage pricing theory of capital market asset pricing, *Journal of Economic Theory* 13, 341–360.
- Whitelaw, Robert F., 1994, Time variations and covariations in the expectations and volatility of stock market returns, *Journal of Finance* XLIX, 515–541.

Appendix A: Proofs

Proof of theorem 1. By definition, the expected rates of returns satisfy $E_0(R_1) = E_0(P_1)/P_0 = 1 + \mu_1$ and $E_0(R_{0:2}) = E_0(R_1R_2) = E_0(\tilde{x})/P_0 = E_0(1 + \mu_2) = (1 + \mu_1)$ where the last equality follows from the steady state assumption.

We need to prove that

$$\text{Cov}_0\left(\frac{1}{x}, y\right) > 0$$

where

$$\begin{aligned} \frac{1}{x} &= \frac{1 + \mu_2}{1 + \mu_1} \\ y &= \frac{E_1(\tilde{x})}{E_0(\tilde{x})}. \end{aligned}$$

Since we have

$$R_1 = \frac{P_1}{P_0} = xyE_0[R_1],$$

we can rewrite it as

$$\frac{1}{x} \frac{R_1}{E_0[R_1]} = y.$$

Substitute y into $\text{Cov}_0(\frac{1}{x}, y)$ to get:

$$\begin{aligned} \text{Cov}_0\left(\frac{1}{x}, y\right) &= \text{Cov}_0\left(\frac{1}{x}, \frac{1}{x} \frac{R_1}{E_0[R_1]}\right) \\ &= E_0\left[\frac{1}{x} \left(\frac{1}{x} \frac{R_1}{E_0[R_1]}\right)\right] - E_0\left(\frac{1}{x}\right)E_0\left(\frac{1}{x} \frac{R_1}{E_0[R_1]}\right) \\ &= E_0\left[\left(\frac{1 + \mu_2}{1 + \mu_1}\right)^2 \frac{R_1}{E_0[R_1]}\right] - E_0\left(\frac{1 + \mu_2}{1 + \mu_1}\right)E_0\left(\frac{1 + \mu_2}{1 + \mu_1} \frac{R_1}{E_0[R_1]}\right). \end{aligned}$$

No leverage, $\text{Cov}_0(\mu_2^2, R_1) = 0$, implies

$$\begin{aligned} E_0\left[\left(\frac{1 + \mu_2}{1 + \mu_1}\right)^2 \frac{R_1}{E_0[R_1]}\right] &= E_0\left[\frac{(1 + \mu_2)^2}{(1 + \mu_1)^2}\right] E_0\left[\frac{R_1}{E_0[R_1]}\right] \\ &= \frac{E_0[(1 + \mu_2)^2]}{(1 + \mu_1)^2} \cdot 1 \\ &> \frac{(E_0[1 + \mu_2])^2}{(1 + \mu_1)^2} \\ &= 1 \end{aligned}$$

Now, it is clear that

$$E_0 \left(\frac{1 + \mu_2}{1 + \mu_1} \right) E_0 \left(\frac{1 + \mu_2}{1 + \mu_1} \frac{R_1}{E_0[R_1]} \right) = E_0 \left(\frac{1 + \mu_2}{1 + \mu_1} \right) E_0 \left(\frac{R_1}{E_0[R_1]} \right) = 1$$

Accordingly,

$$\begin{aligned} \text{Cov}_0 \left(\frac{1}{x}, y \right) &= E_0 \left[\left(\frac{1 + \mu_2}{1 + \mu_1} \right)^2 \frac{R_1}{E_0[R_1]} \right] - E_0 \left(\frac{1 + \mu_2}{1 + \mu_1} \right) E_0 \left(\frac{1 + \mu_2}{1 + \mu_1} \frac{R_1}{E_0[R_1]} \right) \\ &> 1 - 1 \\ &= 0 \end{aligned}$$

□

Proof of Proposition 2. The Euler equation follows directly from the first order conditions the optimal portfolio problem, $\max_{\alpha} E_t u(\tilde{x}_t R_{t:T}(\alpha))$ and the market clearing condition $\alpha = 1$. To prove that the solution is independent of the frequency of trade, consider buying the market at time t and hold it until time s . The value of this claim is

$$\begin{aligned} P_t^* &= \frac{E_t \{u'(\tilde{x}_T) P_s\}}{E_t \{u'(\tilde{x}_T) R_{t:s,f}\}} = E_t \left\{ u'(\tilde{x}_T) \frac{E_s \{u'(\tilde{x}_T) \tilde{x}_T\}}{E_s \{u'(\tilde{x}_T)\} R_{s:T,f}} \right\} \times \frac{1}{E_t \{u'(\tilde{x}_T)\} R_{t:s,f}} \\ &= E_t E_s \left\{ \frac{u'(\tilde{x}_T)}{E_t \{u'(\tilde{x}_T)\}} \frac{E_s \{u'(\tilde{x}_T) \tilde{x}_T\}}{E_s \{u'(\tilde{x}_T)\} R_{t:T,f}} \right\} = E_t \left\{ \frac{E_s \{u'(\tilde{x}_T)\}}{E_t \{u'(\tilde{x}_T)\}} \frac{E_s \{u'(\tilde{x}_T) \tilde{x}_T\}}{E_s \{u'(\tilde{x}_T)\} R_{t:T,f}} \right\} \\ &= E_t \left\{ \frac{E_s \{u'(\tilde{x}_T) \tilde{x}_T\}}{E_t \{u'(\tilde{x}_T)\} R_{t:T,f}} \right\} = \frac{E_t \{u'(\tilde{x}_T) \tilde{x}_T\}}{E_t \{u'(\tilde{x}_T)\} R_{t:T,f}} = P_t. \quad (38) \end{aligned}$$

Since $P_t^* = P_t$ the market value is unchanged by portfolio re-balancing at date s . □

Proof of Proposition 3. By assumption the infinite horizon limit exists, so $\beta(T) \rightarrow \beta$ (constant). □

Proof of Proposition 4. EZ utility is defined as

$$U(\tilde{x}_t, X_t) = \{(1 - \beta) u_t^{1 - \frac{1}{\psi}} + \beta [E_t (U(\tilde{x}_{t+1}, X_{t+1})^{1 - \gamma})]^{1 - \frac{1}{\psi}}\}^{\frac{1}{1 - \frac{1}{\psi}}}$$

In our economy, there is no intermediate consumption, so the period utility $u_t = 0$. If at the same time, we let $\psi \rightarrow \infty$, we can get:

$$U(\tilde{x}_t, X_t) = \beta [E_t(U(\tilde{x}_{t+1}, X_{t+1})^{1-\gamma})]^{\frac{1}{1-\gamma}}$$

Let's redefine $V_t = \frac{U(\tilde{x}_t, X_t)^{1-\gamma}}{1-\gamma}$ and the subjective discount factor $\delta = \beta^{1-\gamma}$:

$$\begin{aligned} V_t &= \delta E_t[V_{t+1}] \\ &= \delta E_t[\delta E_{t+1}[V_{t+2}]] \\ &= \delta^2 E_t[V_{t+2}] \\ &= \dots \\ &= \delta^{T-t} E_t[V_T] \\ &= \delta^{T-t} E_t\left[\frac{U((\tilde{x}_T, X_T)^{1-\gamma})}{1-\gamma}\right] \\ &= \delta^{T-t} E_t\left[\frac{\tilde{x}_T^{1-\gamma}}{1-\gamma}\right]. \end{aligned}$$

As a result, our terminal utility of wealth model is a special case of EZ when $\psi \rightarrow \infty$ and there is no intermediate consumption. □

Deriving λ_σ in section 3.2. The coefficients characterizing the expectations in (24) gives the ODEs

$$\begin{bmatrix} \frac{\partial \hat{\beta}_w}{\partial s} \\ \frac{\partial \hat{\beta}_\sigma}{\partial s} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & -\kappa \end{bmatrix} \begin{bmatrix} \hat{\beta}_w \\ \hat{\beta}_\sigma \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ \beta_w & \beta_\sigma \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \begin{bmatrix} \beta_w \\ \beta_\sigma \end{bmatrix} + \begin{bmatrix} 0 \\ l_1(\varrho(\beta_\sigma) - 1) \end{bmatrix}$$

We get $\beta_w = 1 - \gamma$ and $\beta_w = -\gamma$ respectively. The ODE for β_σ becomes

$$\frac{\partial \beta_\sigma}{\partial s} = -\frac{1}{2}\beta_w - \kappa\beta_\sigma + \frac{1}{2}\beta_w^2 + \sigma_v^2\hat{\beta}_\sigma^2 + l_1(\varrho(\beta_\sigma) - 1), \beta_\sigma(0) = 0. \quad (39)$$

In general, this equation does not admit an analytic solution. The parametric constraints $\sigma_v = l_1 = 0$ give the solution

$$\beta_\sigma(u, s) = (e^{-\kappa s} - 1) \frac{\beta_w(u)(\beta_w(u) - 1)}{2\kappa}. \quad (40)$$

Equation 27 now follows. □

Derivation of the beta model in section 4. The cash flow process and stochastic volatility process of the market portfolio are specified as

$$\begin{aligned}\frac{d\tilde{x}_t}{\tilde{x}_t} &= (\mu_m + \phi_m \sigma_{m,t}^2)dt + \sigma_{m,t}dB_{m,t} \\ d\sigma_{m,t}^2 &= \kappa_m(\theta_m - \sigma_{m,t}^2)dt + \sigma_{v,m}\sigma_{m,t}dZ_{m,t}\end{aligned}$$

The cash flow process and the idiosyncratic stochastic volatility process for asset i are specified as

$$\begin{aligned}\frac{d\tilde{x}_{i,t}}{\tilde{x}_{i,t}} &= (\mu_i + \phi_i \sigma_{m,t}^2)dt + \sigma_{i,t}dB_{i,t} + C_i \sigma_{m,t}dB_{m,t} \\ d\sigma_{i,t}^2 &= \kappa_i(\theta_i - \sigma_{i,t}^2)dt + \sigma_{v,i}\sigma_{i,t}dZ_{i,t}\end{aligned}$$

As a result, the log cash flow processes are

$$\begin{aligned}d \ln \tilde{x}_t &= \left(\mu_m - \frac{1}{2}\sigma_{m,t}^2 + \phi_m \sigma_{m,t}^2\right)dt + \sigma_{m,t}dB_{m,t} \\ d \ln \tilde{x}_{i,t} &= \left(\mu_i - \frac{1}{2}\sigma_{i,t}^2 - \frac{1}{2}C_i^2 \sigma_{m,t}^2 + \phi_i \sigma_{m,t}^2\right)dt + \sigma_{i,t}dB_{i,t} + C_i \sigma_{m,t}dB_{m,t}\end{aligned}$$

We assume that $cov_t(dB_{m,t}, dB_{i,t}) = 0$; $cov_t(dB_{m,t}, dZ_{m,t}) = 0$; $cov_t(dB_{i,t}, dZ_{i,t}) = 0$; $cov_t(dZ_{m,t}, dZ_{i,t}) = 0$.

Let the state variable be $X_t = (\ln \tilde{x}_t, \ln \tilde{x}_{i,t}, \sigma_{m,t}^2, \sigma_{i,t}^2)^T$. Assume that the representative agent has power utility, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$. By the F.O.C, the price of the asset i at time t , $P_{i,t}$ is given as:

$$\begin{aligned}P_{i,t} &= \frac{E_t[\tilde{x}_T^{-\gamma} \tilde{x}_{i,T}]}{E_t[\tilde{x}_{m,T}^{-\gamma}]R_{f,t:T}} \\ &= \frac{E_t[\exp(u_1 X_T)]}{E_t[\exp(u_0 X_T)]R_{f,t:T}} \\ &= \frac{\exp\{\alpha(t, u_1) + \beta^T(t, u_1)X_t\}}{\exp\{\alpha(t, u_0) + \beta^T(t, u_0)X_t\}R_{f,t:T}} \\ &= \frac{1}{R_{f,t:T}} e^{[\alpha(t, u_1) - \alpha(t, u_0)] + [\beta^T(t, u_1) - \beta^T(t, u_0)]X_t} \\ &= \frac{1}{R_{f,t:T}} e^{\lambda_{i,\alpha}(t) + \lambda_{i,\beta}^T(t)X_t}\end{aligned}$$

where the vector $u_1 = (-\gamma, 1, 0, 0)$ and $u_0 = (-\gamma, 0, 0, 0)$. $\lambda_{i,\beta}(t) = [\lambda_{i,\beta_1}(t), \lambda_{i,\beta_2}(t), \lambda_{i,\beta_3}(t), \lambda_{i,\beta_4}(t)]^T$

Similarly, the price of the market portfolio $P_{m,t}$ is

$$\begin{aligned}
P_{m,t} &= \frac{E_t[\tilde{x}_T^{1-\gamma}]}{E_t[\tilde{x}_T^{-\gamma}]R_{f,t:T}} \\
&= \frac{1}{R_{f,t:T}} e^{[\alpha(t,u_m)-\alpha(t,u_0)]+[\beta^T(t,u_m)-\beta^T(t,u_0)]X_t} \\
&= \frac{1}{R_{f,t:T}} e^{\lambda_{m,\alpha}(t)+\lambda_{m,\beta}^T(t)X_t}
\end{aligned}$$

where the vector $u_m = (1 - \gamma, 0, 0, 0)$. $\lambda_{m,\beta}(t) = [\lambda_{m,\beta_1}(t), \lambda_{m,\beta_2}(t), \lambda_{m,\beta_3}(t), \lambda_{m,\beta_4}(t)]^T$

Using similar notation in Duffie et. al. (2000), we summarize the dynamics of state variable X_t as:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t^X$$

where $\mu(X_t)$ and $\sigma(X_t)$ are affine functions of X_t :

$$\begin{aligned}
\mu(X_t) &= K_0 + K_1 X_t \\
(\sigma(X_t)\sigma(X_t)^T)_{ij} &= (H_0)_{ij} + (H_1)_{ij} \cdot X_t
\end{aligned}$$

In our model, we can also write the variance-covariance matrix as

$$\sigma(X_t)\sigma(X_t)^T = H_0 + H_{1,w_m}w_{m,t} + H_{1,w_i}w_{i,t} + H_{1,\sigma_m}\sigma_{m,t}^2 + H_{1,\sigma_i}\sigma_{i,t}^2$$

The matrices are defined as below:

$$K_0 = \begin{bmatrix} \mu_m \\ \mu_i \\ \kappa_m\theta_m \\ \kappa_i\theta_i \end{bmatrix} \quad K_1 = \begin{bmatrix} 0 & 0 & \phi_m - \frac{1}{2} & 0 \\ 0 & 0 & \phi_i - \frac{C_i^2}{2} & -\frac{1}{2} \\ 0 & 0 & -\kappa_m & 0 \\ 0 & 0 & 0 & -\kappa_i \end{bmatrix}$$

$H_0 = 0_{4 \times 4}$, $H_{1,w_m} = 0_{4 \times 4}$, $H_{1,w_i} = 0_{4 \times 4}$ and

$$H_{1,\sigma_m} = \begin{bmatrix} 1 & C_i & 0 & 0 \\ C_i & C_i^2 & 0 & 0 \\ 0 & 0 & \sigma_{v,m}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad H_{1,\sigma_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{v,i}^2 \end{bmatrix}$$

The coefficients $\alpha(t, u)$ and $\beta^T(t, u)$ in the formula of $P_{i,t}$ are solved by the ODEs:

$$\begin{aligned}\dot{\beta} &= -K_1^T \beta - \frac{1}{2} \beta^T \cdot H_1 \cdot \beta \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} - \phi_m & \frac{C_i^2}{2} - \phi_i & \kappa_m & 0 \\ 0 & \frac{1}{2} & 0 & \kappa_i \end{bmatrix} \beta - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \beta^T H_{1,\sigma_m} \beta \\ \beta^T H_{1,\sigma_i} \beta \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\dot{\alpha} &= -K_0^T \beta \\ &= -[\mu_m \quad \mu_i \quad \kappa_m \theta_m \quad \kappa_i \theta_i] \beta\end{aligned}$$

with boundary conditions:

$$\beta(T, u) = u$$

Stationarity of σ_t implies that $\dot{\beta}(t) = 0$ when $T - t \rightarrow \infty$

To solve the coefficients, let's denote the vector $\beta(t, u) = [\beta_{w_m}(t, u), \beta_{w_i}(t, u), \beta_{\sigma_m}(t, u), \beta_{\sigma_i}(t, u)]^T$.

It is clear that

$$\begin{aligned}\dot{\beta}_{w_m}(t, u) &= 0 \\ \dot{\beta}_{w_i}(t, u) &= 0\end{aligned}$$

Let $u = [u(1), u(2), u(3), u(4)]$, then we obtain

$$\begin{aligned}\beta_{w_m}(t, u) &= u(1) \\ \beta_{w_i}(t, u) &= u(2)\end{aligned}$$

$\beta_{\sigma_m}(t, u)$ and $\beta_{\sigma_i}(t, u)$ are solved by two quadratic equations respectively:

$$\begin{aligned}\sigma_{v,m}^2 \beta_{\sigma_m}^2 - 2\kappa_m \beta_{\sigma_m} + \beta_{w_m}^2 + 2C_i \beta_{w_m} \beta_{w_i} - (1 - 2\phi_m) \beta_{w_m} - (C_i^2 - 2\phi_i) \beta_{w_i} + C_i^2 \beta_{w_i}^2 &= 0 \\ \sigma_{v,i}^2 \beta_{\sigma_i}^2 - 2\kappa_i \beta_{\sigma_i} + \beta_{w_i}^2 - \beta_{w_i} &= 0\end{aligned}$$

The solutions of these two quadratic equations are given as:

$$\beta_{\sigma_m}(t, u) = \frac{1}{\sigma_{v,m}^2}(\kappa_m \pm \sqrt{\Delta_m(u)})$$

$$\Delta_m(u) = \kappa_m^2 - \sigma_{v,m}^2[u^2(1) + 2C_1u(1)u(2) - (1 - 2\phi_m)u(1) - (C_i^2 - 2\phi_i)u(2) + C_i^2u(2)^2]$$

$$\beta_{\sigma_i}(t, u) = \frac{1}{\sigma_{v,i}^2}(\kappa_m \pm \sqrt{\Delta_i(u)})$$

$$\Delta_i(u) = \kappa_i^2 - 2\sigma_{v,i}^2[\frac{1}{2}u^2(2) - \frac{1}{2}u(2)]$$

The roots with " + " sign explode when $\sigma_{v,m}$ and/or $\sigma_{v,i}$ goes to zero. We drop these roots because $\beta_{\sigma_m}(t, u)$ and $\beta_{\sigma_i}(t, u)$ are bounded.

Now let's calculate the instantaneous excess return of asset i and market portfolio:

$$\frac{dP_{i,t}}{P_{i,t}} - r_{f,t}dt = [\lambda_{i,\alpha}(t) + \lambda_{i,\beta}^T(t)X_t]dt + \lambda_{i,\beta}^T(t)dX_t + \frac{1}{2}\lambda_{i,\beta}^T(t)(dX_t dX_t^T)\lambda_{i,\beta}(t)$$

$$= -K_0^T \lambda_{i,\beta}(t)dt + \lambda_{i,\beta}^T(t)dX_t + \frac{1}{2}\lambda_{i,\beta}^T(t)(dX_t dX_t^T)\lambda_{i,\beta}(t)$$

$$\frac{dP_{m,t}}{P_{m,t}} - r_{f,t}dt = [\lambda_{m,\alpha}(t) + \lambda_{m,\beta}^T(t)X_t]dt + \lambda_{m,\beta}^T(t)dX_t + \frac{1}{2}\lambda_{m,\beta}^T(t)(dX_t dX_t^T)\lambda_{m,\beta}(t)$$

$$= -K_0^T \lambda_{m,\beta}(t)dt + \lambda_{m,\beta}^T(t)dX_t + \frac{1}{2}\lambda_{m,\beta}^T(t)(dX_t dX_t^T)\lambda_{m,\beta}(t)$$

The second step follows because when $T - t \rightarrow \infty$, $\lambda_{i,\beta}^T(t) = \lambda_{m,\beta}^T(t) = 0$; $\lambda_{i,\alpha}(t) = \dot{\alpha}(t, u_1) - \dot{\alpha}(t, u_0) = -K_0^T(\beta(t, u_1) - \beta(t, u_0)) = -K_0^T \lambda_{i,\beta}(t)$. Similarly, $\lambda_{m,\alpha}(t) = -K_0^T \lambda_{m,\beta}(t)$.

Let's use the formula of $\beta(t, u)$ we obtained above and plug in $u_1 = (-\gamma, 1, 0, 0)$, $u_0 = (-\gamma, 0, 0, 0)$ and $u_m = (1 - \gamma, 0, 0, 0)$ to calculate the expressions of $\lambda_{i,\beta}(t)$ and $\lambda_{m,\beta}(t)$:

$$\begin{aligned} \lambda_{i,\beta}(t) &= \beta(t, u_1) - \beta(t, u_0) \\ &= [0, 1, \frac{1}{\sigma_{v,m}^2}(\sqrt{\Delta_m(u_0)} - \sqrt{\Delta_m(u_1)}), \frac{1}{\sigma_{v,i}^2}(\sqrt{\Delta_i(u_0)} - \sqrt{\Delta_i(u_1)})]^T \\ &= [0, 1, \frac{1}{\sigma_{v,m}^2}(\sqrt{\Delta_m(u_0)} - \sqrt{\Delta_m(u_1)}), 0]^T \\ &= [0, 1, \lambda_i, 0]^T \end{aligned}$$

$$\begin{aligned}
\lambda_{m,\beta}(t) &= \beta(t, u_m) - \beta(t, u_0) \\
&= [1, 0, \frac{1}{\sigma_{v,m}^2}(\sqrt{\Delta_m(u_0)} - \sqrt{\Delta_m(u_m)}), \frac{1}{\sigma_{v,i}^2}(\sqrt{\Delta_i(u_0)} - \sqrt{\Delta_i(u_m)})]^T \\
&= [1, 0, \frac{1}{\sigma_{v,m}^2}(\sqrt{\Delta_m(u_0)} - \sqrt{\Delta_m(u_m)}), 0]^T \\
&= [1, 0, \lambda_m, 0]^T
\end{aligned}$$

Now, let's plug the solutions of $\lambda_{i,\beta}(t)$ and $\lambda_{m,\beta}(t)$ into the instantaneous excess return:

$$\begin{aligned}
\frac{dP_{m,t}}{P_{m,t}} - r_{f,t}dt &= -K_0^T \lambda_{m,\beta}(t)dt + \lambda_{m,\beta}^T(t)dX_t + \frac{1}{2}\lambda_{m,\beta}^T(t)(dX_t dX_t^T)\lambda_{m,\beta}(t) \\
&= (\frac{1}{2}\lambda_m^2 \sigma_{v,m}^2 - \lambda_m \kappa_m)\sigma_{m,t}^2 dt + \sigma_{m,t}dB_{m,t} + \lambda_m \sigma_{v,m}\sigma_{m,t}dZ_{m,t}
\end{aligned}$$

$$\begin{aligned}
\frac{dP_{i,t}}{P_{i,t}} - r_{f,t}dt &= -K_0^T \lambda_{i,\beta}(t)dt + \lambda_{i,\beta}^T(t)dX_t + \frac{1}{2}\lambda_{i,\beta}^T(t)(dX_t dX_t^T)\lambda_{i,\beta}(t) \\
&= (\frac{1}{2}\lambda_i^2 \sigma_{v,m}^2 - \lambda_i \kappa_m)\sigma_{m,t}^2 dt + \sigma_{i,t}dB_{i,t} + C_i \sigma_{m,t}dB_{m,t} + \lambda_i \sigma_{v,m}\sigma_{m,t}dZ_{m,t}
\end{aligned}$$

The expected instantaneous excess return of asset i and the market portfolio are:

$$\begin{aligned}
E_t[\frac{dP_{i,t}}{P_{i,t}} - r_{f,t}dt]/dt &= (\frac{1}{2}\lambda_i^2 \sigma_{v,m}^2 - \lambda_i \kappa_m)\sigma_{m,t}^2 \\
E_t[\frac{dP_{m,t}}{P_{m,t}} - r_{f,t}dt]/dt &= (\frac{1}{2}\lambda_m^2 \sigma_{v,m}^2 - \lambda_m \kappa_m)\sigma_{m,t}^2
\end{aligned}$$

The covariance between the instantaneous return of asset i and the instantaneous return of market portfolio is:

$$Cov_t(\frac{dP_{i,t}}{P_{i,t}} - r_{f,t}, \frac{dP_{m,t}}{P_{m,t}} - r_{f,t}dt)/dt = (C_i + \lambda_i \lambda_m \sigma_{v,m}^2)\sigma_{m,t}^2$$

The variance of instantaneous return of market portfolio is:

$$Var_t[\frac{dP_{m,t}}{P_{m,t}} - r_{f,t}]/dt = (1 + \lambda_m^2 \sigma_{v,m}^2)\sigma_{m,t}^2$$

So the traditional $beta_t$ is equal to:

$$beta_t = \frac{C_i + \lambda_i \lambda_m \sigma_{v,m}^2}{1 + \lambda_m^2 \sigma_{v,m}^2}$$

The formulas for λ_m and λ_i are:

$$\begin{aligned} \lambda_m &= \frac{1}{\sigma_{v,m}^2} (\sqrt{\Delta_m(u_0)} - \sqrt{\Delta_m(u_m)}) \\ &= \frac{1}{\sigma_{v,m}^2} [\sqrt{\kappa_m^2 - \sigma_{v,m}^2 (\gamma^2 + (1 - 2\phi_m)\gamma)} - \sqrt{\kappa_m^2 - \sigma_{v,m}^2 [\gamma^2 - (1 + 2\phi_m)\gamma + 2\phi_m]}] \end{aligned}$$

$$\begin{aligned} \lambda_i &= \frac{1}{\sigma_{v,m}^2} (\sqrt{\Delta_m(u_0)} - \sqrt{\Delta_m(u_1)}) \\ &= \frac{1}{\sigma_{v,m}^2} [\sqrt{\kappa_m^2 - \sigma_{v,m}^2 (\gamma^2 + (1 - 2\phi_m)\gamma)} - \sqrt{\kappa_m^2 - \sigma_{v,m}^2 [\gamma^2 + (1 - 2C_i - 2\phi_m)\gamma + 2\phi_i]}] \end{aligned}$$

Now, we can easily check that CAPM holds when $\sigma_{v,m} \rightarrow 0$ and we set the parameter $\phi_m = \phi_i = 0$:

$$\begin{aligned} \lim_{\sigma_{v,m} \rightarrow 0} \lambda_m &= -\frac{\gamma}{\kappa_m} \\ \lim_{\sigma_{v,m} \rightarrow 0} \lambda_i &= -\frac{C_i \gamma}{\kappa_m} \end{aligned}$$

$$\lim_{\sigma_{v,m} \rightarrow 0} beta_t = C_i$$

$$\begin{aligned} \lim_{\sigma_{v,m} \rightarrow 0} E_t \left[\frac{dP_{i,t}}{P_{i,t}} - r_{f,t} dt \right] / dt &= -\lambda_i \kappa_m \sigma_{m,t}^2 \\ &= C_i \gamma \sigma_{m,t}^2 \end{aligned}$$

$$\begin{aligned} \lim_{\sigma_{v,m} \rightarrow 0} E_t \left[\frac{dP_{m,t}}{P_{m,t}} - r_{f,t} dt \right] / dt &= -\lambda_m \kappa_m \sigma_{m,t}^2 \\ &= \gamma \sigma_{m,t}^2 \end{aligned}$$

As a result, CAPM holds in the sense that

$$\lim_{\sigma_{v,m} \rightarrow 0} E_t \left[\frac{dP_{i,t}}{P_{i,t}} - r_{f,t} dt \right] / dt = \lim_{\sigma_{v,m} \rightarrow 0} \text{beta}_t \times E_t \left[\frac{dP_{m,t}}{P_{m,t}} - r_{f,t} dt \right] / dt$$

Or equivalently,

$$\lim_{\sigma_{v,m} \rightarrow 0} E_t \left[\frac{dP_{i,t}}{P_{i,t}} - r_{f,t} dt \right] / dt = \gamma \lim_{\sigma_{v,m} \rightarrow 0} \text{Cov}_t \left(\frac{dP_{i,t}}{P_{i,t}} - r_{f,t}, \frac{dP_{m,t}}{P_{m,t}} - r_{f,t} dt \right) / dt$$

□

Appendix B: Affine jump diffusions

An affine jump diffusion X_t is a Markov process in $D \in \mathbb{R}^n$ with a stochastic differential equation representation

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t + \xi_t \cdot dN_t. \quad (41)$$

W_t is a standard Brownian motion in \mathbb{R}^n . The term $\xi_t \cdot dN_t$ (element-by-element multiplication) captures independent Poisson jump arrivals of size ξ_t and with arrival intensity $l(X_t)$ and jump size distribution ξ_t on \mathbb{R}^n . The distribution of jump sizes is specified through the Laplace transform $\varrho : \mathbb{C} \rightarrow \mathbb{C}$,

$$Ee^{u\xi} = \varrho(u).$$

which is assumed to be well-defined. In our equilibrium theory, $\varrho(u)$ is assumed to exist for real arguments $u \in \mathbb{R}$ and is interpretable as a moment generating function in that case.

The drift, diffusion and arrival intensities are assumed to be given by

$$\begin{aligned} \mu(X_t) &= \mathcal{M} + \mathcal{K}X_t, \\ \Sigma(X_t)\Sigma(X_t)' &= h + \sum_i H_i X_{t,i}, \\ l(X_t) &= l_0 + l_1 X_t, \end{aligned}$$

for $(\mathcal{M}, \mathcal{K}) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$, $(h, H) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}$, $(l_0, l_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$. For X to be well defined, there are additional joint restrictions on the parameters of the model, which are addressed in Duffie and Kan (1996) and Duffie, Pan, and Singleton (2000). Our notation follows Eraker and Shaliastovich (2008).

The affine jump diffusion framework allow us to compute conditional moment generating function as

$$\phi(u, X_t, s) = E_t \left(e^{u' X_{t+s}} \right). \quad (42)$$

for $u \in \mathbb{R}^n$.

Under appropriate technical regularity conditions (see Duffie, Pan and Singleton (2000)), ϕ is exponential affine in X_t ,

$$\phi(u, X_t, s) = e^{\alpha(u,s) + \beta(u,s)' X_t},$$

where α and β satisfy the ODE's

$$\begin{aligned} \frac{\partial \beta}{\partial s} &= \mathcal{K}' \beta + \frac{1}{2} \beta' H \beta + l_1' (\varrho(\beta) - 1), \\ \frac{\partial \alpha}{\partial s} &= \mathcal{M}' \beta + \frac{1}{2} \beta' h \beta + l_0' (\varrho(\beta) - 1), \end{aligned} \quad (43)$$

subject to boundary conditions $\beta(u, 0) = u, \alpha(u, 0) = 0$. Note that the time argument s is interpretable as a time to maturity, $s = T - t$. A change of variable from s to $T - t$ leads to the equivalent ODEs

$$\begin{aligned} \frac{\partial \beta}{\partial t} &= -\mathcal{K}' \beta - \frac{1}{2} \beta' H \beta - l_1' (\varrho(\hat{\beta}) - 1), \\ \frac{\partial \alpha}{\partial t} &= -\mathcal{M}' \beta - \frac{1}{2} \beta' h \beta - l_0' (\varrho(\beta) - 1), \end{aligned} \quad (44)$$

with boundary conditions $\beta(u, T, T) = u, \alpha(u, T, T) = 0$.