

A Closed-form Solution for Outperformance Options with Stochastic Correlation and Stochastic Volatility

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Abstract

This article introduces a multi-asset model based on Wishart processes that accounts for stochastic volatility and for stochastic correlations between the assets returns, as well as between their volatilities. The model accounts for the existence of correlation term structure and allows for a positive link between the assets returns correlation and their volatility that is consistent with the empirical evidence. In this sense, the model is able to account for the existence of correlation skew.

Another advantage of the model is its analytical tractability: it is possible to derive semi-closed form solutions for European options on each underlying asset, as well as for the price of multi-asset exotic options such as the outperformance option. The article shows that the price of this option depends crucially on the term structure of the correlation corresponding to the assets returns. To motivate the introduction of the model, I compare the prices obtained under this model and under other models with constant correlations commonly used by financial institutions. The empirical results show that the consideration of stochastic correlation is a key element for the correct valuation of multi-asset exotic options.

Keywords: Wishart process, stochastic volatility, stochastic correlation, correlation skew, correlation term structure, multi-asset options.

JEL: G12, G13.

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1 Introduction

In recent years there has been a remarkable growth of multi-asset options. These options exhibit sensitivity to the volatility of the underlying assets, as well as to their correlations. Empirical evidence shows that volatility, far from remain static through time, evolves stochastically. This stylized fact can be found in Franks and Schwartz (1991), Avellaneda and Zhu (1997), Derman (1999), Bakshi, Cao and Chen (2000), Cont and da Fonseca (2002), Daglish et al. (2007) and Carr and Wu (2009). Examples of one factor option pricing models that account for this effect can be found in Hull and White (1987) and Heston (1993). On the other hand, da Fonseca et al. (2008) introduce a multifactor volatility model for a single risky asset.

Ball and Torous (2000) show that the correlations between financial quantities changes over time and are notoriously unstable. Moreover, Solnik et al. (1996), as well as Loretan and English (2000) present evidence for the existence of a link between correlation and volatility: correlations between financial assets tend to be high in periods of high market volatility. But, unfortunately, multi-asset option are usually priced assuming constant instantaneous correlations between assets¹. Reghai (2010) states that during a crisis the correlation becomes highly negatively correlated with the stock market. In this sense, Langnau (2009) postulates that in 2008 the crisis in the credit market led to a spike in correlations that resulted in large losses for several exotic trading desks corresponding to different financial institutions. Importantly, this spike in correlation coincided with an increase in the level of volatility. Actually, in November 2008 the VIX index reached its maximum historical closing level. Note that the squared of the VIX index approximates

¹See Krekel et al. (2004) for a comparison of different pricing methods for basket options.

the variance swap rate or delivery price of a variance swap, obtained from the European options corresponding to the Standard and Poor's 500 index with maturity within one month and, therefore, this index provides a measure of the market volatility². In this sense, as Langnau (2009) points out, empirical evidence strongly suggests higher correlations on the market downside giving rise to a correlation skew similar to the implied volatility skew associated with options on single assets. Langnau (2009) and Reghai (2010) introduce a deterministic local correlation model that attends to capture the correlation structure associated with the pricing of multi-asset options.

The outperformance option is a particular case of multi-asset option that exhibits high sensitivity to the correlation between the underlying assets. Consider two assets: asset 1 and asset 2. A European outperformance option is a capital guaranteed structure that pays the positive excess return of asset 1 over asset 2 at the maturity of the option. Therefore, this option allows investors to benefit from a view on the relative performance of two underlying assets without taking any directional exposure to the evolution of the market. Under the assumption that both assets prices follow a Geometric Brownian motion with constant instantaneous volatilities and under the assumption of a constant instantaneous correlation between both underlying assets, Margrabe (1978), Fischer (1978) and Derman (1996), develop closed-form expressions to price European outperformance options.

This article introduces a multi-asset model based on Wishart processes that accounts for stochastic volatility and for stochastic correlation between the underlying assets, as well as between their volatilities. The Wishart process was mathematically developed in Bru (1991) and was introduced

²For a definition of the methodology and the history of the VIX index, see CBOE (2009) and Carr and Wu (2006).

into finance by [Gourieroux and Sufana \(2004\)](#). In their model the Wishart process describes the dynamics of the covariance matrix and it is assumed to be independent of the assets noises. On the other hand, [da Fonseca et al. \(2008\)](#) introduce a correlation structure between the single asset noise and the volatility factors. [Branger and Muck \(2009\)](#) rely on Wishart processes in the pricing of quanto derivatives. In the context of Wishart processes, [Leippold and Trojani \(2010\)](#) allow for jumps and apply their model to the pricing of single asset derivatives, as well as to a multi-asset allocation problem.

The model presented in this article can capture the term structure of correlation and it is able to generate a link between assets returns correlations and their volatilities that is consistent with the empirical evidence. Hence, the model is able to generate positive correlated paths for the correlation and the volatilities corresponding to the assets returns and, in this sense, it is able to account for the existence of correlation skew.

Under the assumptions of the model, using the transform pricing approach for affine processes from [Duffie et al. \(2000\)](#), I derive semi-closed-form solutions for European options on each underlying asset, as well as for the price of European outperformance options. In the context of constant and/or deterministic correlations it is well known that the outperformance option is quite sensitive to the correlation between the underlying assets returns. Importantly, in this article I show that the price of this option depends crucially on the term structure of the correlation corresponding to the evolution of the underlying assets. I then provide a numerical illustration that motivates the introduction of the model comparing the prices obtained under this model and under other models with constant correlations commonly used by financial institutions. In particular, the comparison of the prices obtained under a multi-asset local volatility model and under a multi-asset version of the

Heston (1993) stochastic volatility model with constant instantaneous correlations, shows that these models are not flexible enough to capture the sensitivity of the outperformance option with respect to the term structure of correlation and, therefore, are not able to yield the same price for this option that the model presented in this article.

The features of the model presented in this article make it suitable for pricing other multi-asset options such as best-of or worst-of options which are quite sensitive to the evolution of the correlation corresponding to the underlying assets returns, as well as to the evolution of their instantaneous volatilities.

The rest of the paper proceeds as follows. Section 2 presents the multi-asset Wishart volatility model and the correlation structure generated by the model. This section also provides a numerical example which shows that the model is able to generate a correlation between assets returns that tends to be high in periods of high volatility and, hence, it is consistent with the empirical evidence. Section 3 solves the pricing problem and provides semi-closed-form formulas for the price of standard European options on single assets, as well as for the price of European outperformance options. Pricing formulas corresponding to other multi-asset options, such as digital outperformance options or chooser options are provided as well. This section also offers a numerical analysis of the sensitivity of the outperformance option with respect to the term structure of correlation that motivates the use of a stochastic correlation framework. Section 4 provides a numerical illustration which shows the advantages carried by the Wishart specification with respect to other multi-asset models with constant correlations, such as the local volatility model and a multi-asset version of the Heston (1993) model. Finally, section 5 offers concluding remarks.

2 The multi-asset Wishart volatility model

2.1 Model specification

This section describes the joint dynamics of asset prices. I restrict the analysis to the case of two assets but the model can easily be generalized to consider $n > 2$ assets.

Let $\{S_t := (S_{1t}, S_{2t})' \in \mathbb{R}^2 \ t \geq 0\}$ be the price process of the assets and let us denote by $Y_t = (\ln S_{1t}, \ln S_{2t})'$ the log-return vector. For simplicity, I assume that the continuously compounded risk-free rate r and dividend yield q_i , for $i = 1, 2$, are constant. Let Θ denote the probability measure defined on a probability space $(\Lambda, \mathcal{F}, \Theta)$ such that asset prices expressed in terms of the current account are martingales. We denote this probability measure as the risk-neutral measure. I assume the following dynamics for the return process Y_t under Θ :

$$dY_t = \left[r\mathbf{1} - q - \frac{1}{2} \text{diag}(X_t) \right] dt + \sqrt{X_t} dZ_t \quad (1)$$

where $\mathbf{1}$ is a 2×1 vector of ones, $q = (q_1, q_2)'$ and the vector $\text{diag}(X_t) \in \mathbb{R}^2$ has the i -th component equal to X_{iit} . The matrix X_t belongs to the set of symmetric 2×2 positive semi-definite matrices and satisfies the following stochastic differential equation:

$$dX_t = \left[\Omega \Omega' + M X_t + X_t M' \right] dt + \sqrt{X_t} dB_t Q + Q' dB_t' \sqrt{X_t} \quad (2)$$

where $\Omega, M, Q \in \mathbb{R}^{2 \times 2}$ and B_t is a matrix of standard Brownian motions in $\mathbb{R}^{2 \times 2}$. Equation (2) characterizes the Wishart process introduced by Bru (1991) and represents the matrix analogue of the square root mean-reverting process. In order to grant that X_t is positive semi-definite and the typical mean reverting feature of the volatility, the matrix M is assumed to be

negative semi-definite, while Ω satisfies

$$\Omega\Omega' = \beta Q'Q \quad (3)$$

with the real parameter $\beta > 1$. This condition ensures that X is positive semi-definite at every point in time if X_0 is³.

In equation (1), $Z \in \mathbb{R}^2$ is a standard Brownian motion of the form:

$$Z_t = B_t\rho + \sqrt{1 - \rho'\rho}W_t$$

where $W \in \mathbb{R}^2$ is another standard Brownian motion independent of B and $\rho = (\rho_1, \rho_2)'$ is a fixed correlation vector that determines the correlation between the returns and the state variables. This correlation vector satisfies $\rho'\rho \leq 1$.

Note that the matrix Q characterizes the randomness of the state variable X . The mean reversion level \bar{X} , associated with this variable, is given by the following expression:

$$\Omega\Omega' + M\bar{X} + \bar{X}M' = 0 \quad (4)$$

Taking into account the previous equation, it is possible to rewrite equation (2) as follows:

$$dX_t = \left[-M(\bar{X} - X_t) - (\bar{X} - X_t)M' \right] dt + \sqrt{X_t}dB_tQ + Q'dB'_t\sqrt{X_t}$$

As pointed out by Branger and Muck (2009), the previous equation shows that the matrix M can be considered as the negative of the mean reversion speed corresponding to the volatility process. Note that this matrix is assumed to be negative semi-definite.

³In the general case where we have n assets, this condition becomes $\beta > n - 1$.

2.2 Variance-covariance structure

The model presented in this article, based on Wishart processes, allows us to capture stochastic volatility, stochastic correlation between assets returns, as well as stochastic correlation between volatilities and between volatilities and assets returns.

The conditional covariance matrix of the return process is:

$$V_t := \frac{1}{dt} \text{Var} (dY_t) = X_t$$

whereas the correlation between the assets returns is given by:

$$\rho_{(t)S_1S_2} := \text{Corr} (dY_{1t}, dY_{2t}) = \frac{X_{12t}}{\sqrt{X_{11t}X_{22t}}} \quad (5)$$

Importantly, the Wishart specification allows us to introduce stochastic correlation between assets returns and, as we will see, it can capture several patterns corresponding to the term structure of correlation.

On the other hand, the correlation between each asset return and its own instantaneous variance is:

$$\rho_{S_iV_{ii}} := \text{Corr} (dY_{it}, dV_{iit}) = \frac{\rho_1 Q_{1i} + \rho_2 Q_{2i}}{\sqrt{Q_{i1}^2 + Q_{i2}^2}} \quad i = 1, 2.$$

This correlation is constant⁴ and depends on the correlation vector ρ , as well as on the matrix Q that characterizes the randomness of the state variable X . In this sense, like in the Heston (1993) model, a negative correlation ρ between the process associated with the assets returns and the process corresponding to the variance matrix X leads to a negative implied volatility skew. Note that the existence of an implied volatility skew has characterized equity options markets since the stock market crash on October 1987.

⁴Note that it is easy to make this correlation stochastic by introducing an additional Wishart process as in Branger and Muck (2009). Nevertheless, this effect has the cost of increasing considerably the number of parameters making more difficult the calibration of the model to real market data.

In this model the correlation between each variance process is stochastic and it is given by:

$$\begin{aligned} \rho_{(t)V_{11}V_{22}} &:= \text{Corr}(dV_{11t}, dV_{22t}) = \\ &= \frac{X_{11t}(Q_{21}^2 + Q_{22}^2) + 2X_{12t}(Q_{11}Q_{21} + Q_{12}Q_{22}) + X_{22t}(Q_{11}^2 + Q_{12}^2)}{4\sqrt{X_{11t}X_{22t}(Q_{11}^2 + Q_{12}^2)(Q_{21}^2 + Q_{22}^2)}} = \\ &= \frac{X_{11t}(Q_{21}^2 + Q_{22}^2) + X_{22t}(Q_{11}^2 + Q_{12}^2)}{4\sqrt{X_{11t}X_{22t}(Q_{11}^2 + Q_{12}^2)(Q_{21}^2 + Q_{22}^2)}} + \frac{\rho_{(t)S_1S_2}(Q_{11}Q_{21} + Q_{12}Q_{22})}{2\sqrt{(Q_{11}^2 + Q_{12}^2)(Q_{21}^2 + Q_{22}^2)}} \end{aligned}$$

This correlation is a function of the state variable X and of the elements of Q . Finally, the cross-asset-variance correlations are also stochastic and can be expressed as:

$$\begin{aligned} \rho_{(t)S_1V_{22}} &:= \text{Corr}(dY_{1t}, dV_{22t}) = \rho_{(t)S_1S_2} \frac{(\rho_1 Q_{12} + \rho_2 Q_{22})}{\sqrt{(Q_{21}^2 + Q_{22}^2)}} = \rho_{(t)S_1S_2} \rho_{S_2V_{22}} \\ \rho_{(t)S_2V_{11}} &:= \text{Corr}(dY_{2t}, dV_{11t}) = \rho_{(t)S_1S_2} \frac{(\rho_1 Q_{11} + \rho_2 Q_{21})}{\sqrt{(Q_{11}^2 + Q_{12}^2)}} = \rho_{(t)S_1S_2} \rho_{S_1V_{11}} \end{aligned}$$

These correlations are proportional to the instantaneous correlation between assets returns and between assets returns and their instantaneous variances. Note that the expressions corresponding to the cross-asset-variance correlations are consistent with the method proposed by Jäckel and Kahl (2009) to obtain positive semi-definite correlation matrices in a multi-asset framework with stochastic volatility.

The previous equations show that the model provides quite flexible dynamics corresponding to the term structure of the correlations associated with the assets returns, as well as with their instantaneous variances.

2.3 The link between the correlation of assets returns and their instantaneous variances

As said previously, in the multi-asset Wishart volatility model presented in this article, like in the Heston (1993) model, a negative correlation between the process associated with the assets returns and the process corresponding to the variance matrix X leads to the existence of a negative implied volatility skew.

Table 1: Parameters specification in a bivariate Wishart volatility model for assets returns

| Parameter | Value |
|-----------|---------|
| M_{11} | -2.0300 |
| M_{12} | 0.0000 |
| M_{21} | 0.0000 |
| M_{22} | -2.0300 |
| X_{110} | 0.0484 |
| X_{120} | 0.0350 |
| X_{210} | 0.0350 |
| X_{220} | 0.0426 |
| Q_{11} | 0.4000 |
| Q_{12} | 0.1200 |
| Q_{21} | 0.1200 |
| Q_{22} | 0.3500 |
| ρ_1 | -0.7000 |
| ρ_2 | -0.6500 |
| β | 1.9703 |

Note. X_0 represents the initial level associated with the covariance matrix.

Importantly, the correlation between assets returns implied by the model is stochastic and it is consistent with the existence of a correlation skew suggested by the empirical evidence. In this sense, this section provides a numerical example of the positive link between the correlation of assets returns and their instantaneous volatilities that it is possible to achieve under

the assumptions of the model. To this end, I consider the parametric specification of table 1 and I use Monte Carlo simulations⁵ to generate paths for the covariance matrix X based on the dynamics of equation (2) and I use equation (5) to characterize the time evolution of the correlation between assets returns corresponding to each simulation.

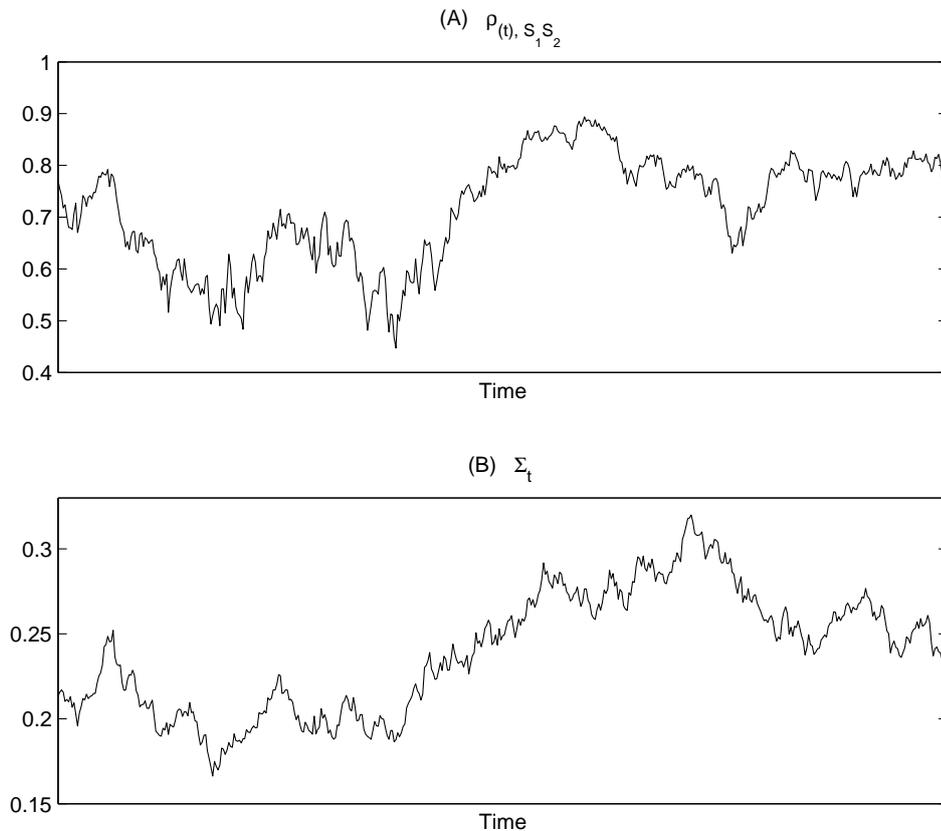


Figure 1: Simulated trajectories associated with the dynamics of the assets returns correlation $\rho_{(t), S_1 S_2}$ (panel A) and the dynamics of the average volatility Σ_t (panel B). The trajectories are simulated using the parametric specification of table 1.

⁵I consider 500 time steps and 1.000 trials.

Let us define the time t average volatility as:

$$\Sigma_t := \frac{\sqrt{X_{11t}} + \sqrt{X_{22t}}}{2} \quad (6)$$

Note that, from table 1, the elements of the correlation vector ρ are negative leading to a negative implied volatility skew for each underlying asset. Taking into account equation (5), the initial level corresponding to the instantaneous correlation between assets returns is 0.7708 whereas, from equation (6), the initial average volatility is given by 0.2132.

Figure 1 provides a simulated path corresponding to the dynamics of the average volatility Σ_t and the correlation $\rho_{(t)S_1S_2}$. The figure shows a positive link between correlation and volatility under the multi-asset Wishart volatility model. In this sense, the correlation tends to be high in periods of high volatility. To analyze formally the link between assets correlation and their instantaneous volatilities within the model, I consider a regression analysis. To this end, I postulate the following relationship between the assets returns correlation and the average instantaneous volatility:

$$\begin{aligned} E_t \left[\rho_{(t)S_1S_2}^i \right] &= 2\varphi \left(\alpha + \eta \Sigma_t^i \right) - 1 \\ \rho_{(t)S_1S_2}^i &= E_t \left[\rho_{(t)S_1S_2}^i \right] + \varepsilon_t^i \\ \varepsilon_{t|t}^i &\sim N(0, \xi_\varepsilon) \end{aligned} \quad (7)$$

where $\rho_{(t)S_1S_2}^i$, Σ_t^i and ε_t^i denotes, respectively, the correlation, the volatility and the error term associated with simulation i in period t . On the other hand, $E_t[\cdot]$ represents the expectation given the information obtained until period t and $\varphi(\cdot)$ is the logistic function:

$$\varphi(x) = \frac{e^x}{1 + e^x}$$

Note that the choice of this functional specification ensures that the estimated correlation lies within $[-1, 1]$. I assume that the innovation follows a normal

distribution conditional on the information available until date t . Under the specification of equation (7) the sensitivity of the expected correlation with respect to the average volatility is given by:

$$\frac{\partial E_t [\rho_{(t)S_1S_2}]}{\partial \Sigma_t} = 2\eta\varphi(\alpha + \eta\Sigma_t^i) [1 - \varphi(\alpha + \eta\Sigma_t^i)] \quad (8)$$

The sign of the previous expression is given by the sign of η . Therefore, if the model implies a positive link between the instantaneous correlation and the volatility, the sign of the estimated coefficient corresponding to η should be greater than zero. Let $\omega = (\alpha, \eta, \xi_\varepsilon)'$ denote the parameter vector and let $L(\omega)$ denote the log-likelihood function evaluated at the true parameter vector. This function takes the following form:

$$L(\omega) = \sum_{i=1}^N \sum_{t=1}^T \ln \left[\frac{1}{\xi_\varepsilon} \phi \left(\frac{\rho_{(t)S_1S_2}^i - 2\varphi(\alpha + \eta\Sigma_t^i) + 1}{\xi_\varepsilon} \right) \right] \quad (9)$$

where N is the total number of simulations considered, T is the total number of periods and ϕ is the standard normal density function. Table 2 shows the maximum likelihood estimators, as well as its standard errors, obtained from the numerical optimization of the conditional log-likelihood function of equation (9). The convergence to the maximum values reported in the table is robust with respect to a broad range of start-up conditions.

Table 2: The link between assets correlation and volatility. Estimation results

| Dependent variable: $\rho_{(t)S_1S_2}^i$ | | | |
|--|------------|----------------|----------------|
| Number of observations: 500,000 | | | |
| | Estimation | Standard error | <i>p-value</i> |
| α | 0.0848 | 0.0029 | 0.0000 |
| η | 7.9570 | 0.0174 | 0.0000 |
| ξ_ε | 0.3075 | 0.0002 | 0.0000 |

Note. The table shows the maximum likelihood estimators corresponding to equation (9).

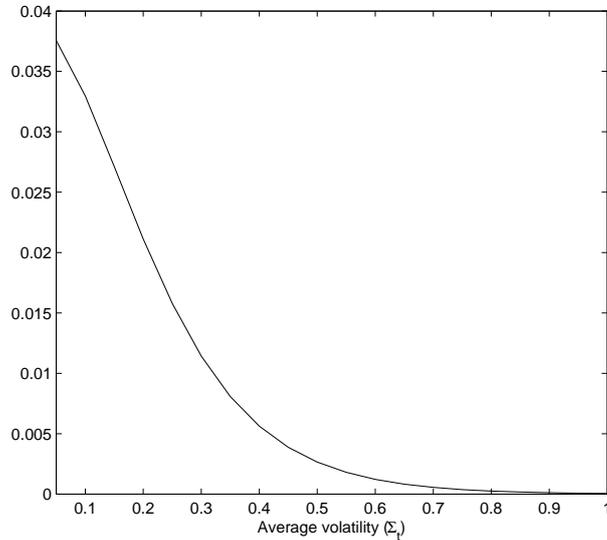


Figure 2: Estimated sensitivity of the assets returns correlation as a function of the average volatility Σ_t corresponding to a 1% change in this variable based on equation (8).

All the estimated parameters are significantly different from zero. Note that the estimated value for η is positive indicating a positive link between the correlation of assets returns and the average level of volatility. As said previously, empirical evidence strongly suggests higher correlations and volatilities on the market downside giving rise to a correlation skew similar to the implied volatility skew associated with options on single assets. Importantly, the previous example shows that the Wishart volatility model is able to replicate this pattern of behavior. This fact is quite relevant given that, as pointed out by Derman and Wilmott (2009), the most important question about any financial model is how useful it is despite its assumptions. Therefore, for the correct pricing of multi-asset options sensitive to the instantaneous correlations and volatilities, it is crucial the development of realistic valuation models that correctly capture the dynamics of these variables.

Finally, figure 2 reports the estimated sensitivity of the assets returns correlation with respect to each 1% of change in the average volatility associated with different levels corresponding to the average volatility, based on equation (8). The figure shows that when both the correlation and the instantaneous volatility are low, the sensitivity is higher. Conversely, when both variables are quite high, an additional increase in the volatility level has less impact on the assets returns correlation.

3 The pricing problem

This section is devoted to the valuation of options in the two-asset framework. I first consider plain vanilla options on each underlying asset. I then consider the outperformance option, which is very sensitive to the correlation between assets returns. To this end, I follow Duffie et al. (2000) to calculate option prices via Fourier inversion.

3.1 The Laplace transform of the assets returns

Following Duffie et al. (2000), given some $(Y, T, a, b) \in \mathbb{R}^2 \times [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$, let $G_{a,b}(y; Y_0, T) : \mathbb{R} \rightarrow \mathbb{R}_+$ denote the price of a security that pays e^{aY_T} at time $t = T$ in the event that $bY_T \leq y$:

$$G_{a,b}(y; Y_0, T) := E_{\Theta} \left[e^{-\int_0^T R(s) ds} e^{aY_T} \mathbf{1}_{(bY_T \leq y)} \right] \quad (10)$$

where $\mathbf{1}_{(bY_T \leq y)}$ is the Heaviside step function or unit step function and where $R(s)$ denotes the instantaneous short-term interest rate. As said previously, for simplicity, I assume that it is constant so that $R(s) = r$. Along the lines of Duffie et al. (2000), the Fourier transform of $G_{a,b}(y; Y_0, T)$ is given by:

$$\Psi_{a,b}(z; Y_0, T) = \int_{-\infty}^{\infty} e^{izy} dG_{a,b}(y; Y_0, T) = E_{\Theta} \left[e^{-rT} e^{\gamma' Y_T} \right]$$

where $z \in \mathbb{R}$, $i^2 = -1$ and $\gamma = a + bzi$. Duffie et al. (2000) show that the function $G_{a,b}(y; Y_0, T)$ can be calculated via the following inversion formula:

$$G_{a,b}(y; Y_0, T) = \frac{\Psi_{a,b}(0; Y_0, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\Psi_{a,b}(v; Y_0, T) e^{-ivy}]}{v} dv \quad (11)$$

where $Im(c)$ denotes the imaginary part of $c \in \mathbb{C}$. Appendix A shows that under the risk-neutral measure Θ , the Fourier transform $\Psi_{a,b}(z; Y_0, T)$ is given by:

$$\Psi_{a,b}(z; Y_0, T) = e^{B(\gamma, T) + \text{tr}[A(\gamma, T)X_0] + \gamma'Y_0}$$

The matrix $A(\gamma, T)$ can be expressed as follows:

$$A(\gamma, T) = C_{22}^{-1}(\gamma, T) C_{21}(\gamma, T)$$

where:

$$\begin{pmatrix} C_{11}(\gamma, T) & C_{12}(\gamma, T) \\ C_{21}(\gamma, T) & C_{22}(\gamma, T) \end{pmatrix} = \exp \left[\begin{pmatrix} M + Q'\rho\gamma' & -2Q'Q \\ d_1(\gamma) & -M' - \gamma\rho'Q \end{pmatrix} T \right]$$

$$d_1(\gamma) = \frac{1}{2} [\gamma\gamma' - \text{diag}(\gamma)]$$

On the other hand, $B(\gamma, T)$ is given by:

$$B(\gamma, T) = d_0(\gamma)T - \frac{\beta}{2} \text{tr}[\ln[C_{22}(\gamma, T)] + T(M + Q'\rho\gamma)']$$

where:

$$d_0(\gamma) = \sum_{i=1}^2 \gamma_i(r - q_i) - r$$

Taking into account the previous expressions, it is possible to obtain semi-closed-form solutions for European options based on the numerical integration of equation (11).

3.2 Pricing plain vanilla options on each underlying asset

In this section I deal with the pricing problem of plain vanilla contingent claims. In particular, let us consider a European call on asset 1 with strike

K and payoff $(S_{1T} - K)^+$ and let e_i , $i = 1, 2$, denotes the i -th unit vector in \mathbb{R}^2 so that $e_1 = (1, 0)$ and $e_2 = (0, 1)$. In this case, it is possible to express the payoff associated with the European call as follows:

$$(S_{1T} - K)^+ = e^{Y_{1T}} \mathbf{1}_{(-Y_{1T} \leq -\ln K)} - K \mathbf{1}_{(-Y_{1T} \leq -\ln K)}$$

Taking into account the definition of $G_{a,b}(y; Y_0, T)$ in equation (10), it is possible to express the time $t = 0$ price of the European call on asset 1 with maturity $t = T$ as:

$$C(S_{10}, T, K) = G_{e_1, -e_1}(-\ln K; Y_0, T) - KG_{0, -e_1}(-\ln K; Y_0, T)$$

Analogously, the time $t = 0$ price of the European call on asset 2 is given by:

$$C(S_{20}, T, K) = G_{e_2, -e_2}(-\ln K; Y_0, T) - KG_{0, -e_2}(-\ln K; Y_0, T)$$

3.3 The outperformance option

As said in the introductory section, the outperformance option is a particular case of multi-asset option that exhibits high sensitivity to the correlation between the underlying assets. Its payoff is given by:

$$(S_{1T} - S_{2T})^+$$

The lower the correlation between assets returns, the higher the probability of obtaining a high payoff and, therefore, the premium of the outperformance option exhibits negative sensitivity to the correlation between assets returns. Let us assume that both assets prices follow a Geometric Brownian motion with constant instantaneous volatilities σ_i , $i = 1, 2$, and with a constant instantaneous correlation λ between the assets returns corresponding to both underlying assets. I denote this specification as the multi-asset Black-Scholes

(1973) (MBS) model. In this case, it is well known⁶ that the time $t = 0$ price of the outperformance option with maturity $t = T$, under the risk-neutral probability measure Θ , is given by:

$$O_{0,T} := e^{-rT} E_{\Theta} \left[(S_{1T} - S_{2T})^+ \right] = S_{10} e^{-q_1 T} \Phi(h) - S_{20} e^{-q_2 T} \Phi(h - \delta \sqrt{T}) \quad (12)$$

where Φ is the standard normal cumulative distribution function and where h and δ are given by:

$$h = \frac{1}{\delta \sqrt{T}} \left[\ln \left(\frac{S_{10}}{S_{20}} \right) + \frac{\delta^2}{2} T + (q_2 - q_1) T \right]$$

$$\delta = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \lambda}$$

The bivariate Wishart volatility model preserves the analytical tractability and allows for the existence of stochastic correlation and stochastic volatility that, as we will see, are determinant elements for the correct valuation of the outperformance option. In this sense, it is possible to express the payoff corresponding to the outperformance option as follows:

$$(S_{1T} - S_{2T})^+ = (S_{1T} - S_{2T}) \mathbf{1}_{(S_{1T} \geq S_{2T})} = e^{Y_{1T}} \mathbf{1}_{((e_2 - e_1) Y_T \leq 0)} - e^{Y_{2T}} \mathbf{1}_{((e_2 - e_1) Y_T \leq 0)}$$

Hence, taking into account the definition of $G_{a,b}(y; Y_0, T)$ in equation (10), the time $t = 0$ price of the European outperformance option is:

$$O_{0,T} = G_{e_1, e_2 - e_1}(0; Y_0, T) - G_{e_2, e_2 - e_1}(0; Y_0, T) \quad (13)$$

Under the assumption of the Wishart volatility model it is also possible to obtain semi-closed-form solutions for the prices of other exotic options. For example, let us consider a European digital outperformance option that pays a certain percentage of the option nominal, denoted as option coupon, if the

⁶See, for example, Margrabe (1978), Fischer (1978) and Derman (1996).

performance of asset 1 is above the performance of asset 2 at the maturity of the option. Marabel (2011) shows that the cross-gamma corresponding to this option changes sign depending on the relative evolution of both underlying assets. The author shows that it can be dangerous to price options, with a cross-gamma that changes sign, assuming a constant instantaneous correlation. Under the assumptions of the multi-asset Wishart volatility model we can express the price of a European digital outperformance option as follows:

$$OD_{0T} := e^{-rT} E_{\Theta} [\mathbf{1}_{(S_{1T} > S_{2T})}] = G_{0, e_2 - e_1} (0; Y_0, T)$$

Another structure related to the outperformance option is the forward on the best performing stock⁷, also known as chooser option, where the payoff at the maturity of the structure is given by:

$$\max (S_{1T}, S_{2T})$$

Under the assumptions of the Wishart volatility model we have the following analytic formula corresponding to the time $t = 0$ price of the forward on the best performing stock with maturity $t = T$:

$$FBO_{0T} := e^{-rT} E_{\Theta} [\max (S_{1T}, S_{2T})] = G_{e_1, e_2 - e_1} (0; Y_0, T) + G_{e_2, e_1 - e_2} (0; Y_0, T)$$

On the other hand, the payoff associated with the forward on the worst performing stock can be expressed as:

$$\min (S_{1T}, S_{2T})$$

and the corresponding time $t = 0$ premium is given by:

$$FWO_{0T} := e^{-rT} E_{\Theta} [\min (S_{1T}, S_{2T})] = G_{e_1, e_1 - e_2} (0; Y_0, T) + G_{e_2, e_2 - e_1} (0; Y_0, T)$$

⁷Note that the payoff of the outperformance option can also be expressed as:

$$(S_{1T} - S_{2T})^+ = \max (S_{1T}, S_{2T}) - S_{2T}$$

The outperformance option, as well as the forward on the best and on the worst performing stock, like other multi-asset exotic options, are mainly quoted in the over-the-counter markets. These markets are relatively liquid for some equity indexes such as the Eurostoxx 50 index, the Standard and Poor's 500 index and the Nikkei 225 index. But, unfortunately, they are less liquid for other indexes and for single stocks. Given the analytical tractability of the previous options under the multi-asset Wishart volatility model, the development of these markets will make it possible to use the market prices corresponding to these exotic options together with the market prices associated with plain vanilla options to jointly calibrate the parameters of the model to real market data. This is a very important feature of the model, given that it is possible to use exotic options as instruments in the calibration process and not only plain vanilla options. Note that the consideration of plain vanilla options only, on the calibration process, is the usual situation in other models such as the local volatility model or the Heston (1993) model.

3.3.1 The term structure of correlation and the outperformance option

This section provides a numerical example of the sensitivity of the outperformance option with respect to the term structure of correlation that motivates the use of a stochastic correlation framework for the correct valuation of multi-asset options that exhibit sensitivity to this variable.

Let us consider the parametric specification of table 1. I also assume that the continuously compounded risk-free rate r is equal to 1.5% and dividend yield q_i , is equal to 3% for $i = 1, 2$. Table 3 shows the implied volatility surface generated by this parameter set for asset 1, whereas table 4 provides

the implied volatility surface corresponding to asset 2. On the other hand, figure 3 shows the graphical representation associated with both volatility surfaces. In both cases the figure reveals the existence a of negative volatility skew, which is most pronounced for near-term options and it is consistent with the negative sign of the elements corresponding to the correlation vector ρ . This is a common pattern of behavior that has been widely observed in the equity options market. Some examples can be found, among others, in Derman et al. (1995) and Gatheral (2006).

Table 3: Implied volatility surface for asset 1 generated by the bivariate Wishart volatility model.

| T/K | 70% | 80% | 90% | 95% | 100% | 105% | 110% | 120% | 130% |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.50 | 34.36 | 30.83 | 27.33 | 25.58 | 23.81 | 22.04 | 20.32 | 17.19 | 15.12 |
| 0.75 | 33.03 | 30.07 | 27.23 | 25.83 | 24.47 | 23.13 | 21.82 | 19.35 | 17.23 |
| 1.00 | 32.03 | 29.60 | 27.21 | 26.03 | 24.90 | 23.83 | 22.80 | 20.85 | 18.97 |
| 1.25 | 31.52 | 29.17 | 27.11 | 26.18 | 25.28 | 24.42 | 23.57 | 21.92 | 20.27 |
| 1.50 | 30.74 | 28.83 | 27.19 | 26.41 | 25.65 | 24.89 | 24.15 | 22.68 | 21.23 |
| 1.75 | 30.27 | 28.75 | 27.33 | 26.63 | 25.94 | 25.26 | 24.58 | 23.27 | 21.97 |
| 2.00 | 30.08 | 28.75 | 27.44 | 26.80 | 26.17 | 25.54 | 24.92 | 23.73 | 22.56 |
| 2.25 | 30.01 | 28.77 | 27.54 | 26.93 | 26.34 | 25.76 | 25.19 | 24.09 | 23.03 |
| 2.50 | 29.97 | 28.77 | 27.60 | 27.03 | 26.48 | 25.94 | 25.41 | 24.41 | 23.41 |
| 2.75 | 29.93 | 28.77 | 27.65 | 27.11 | 26.58 | 26.09 | 25.59 | 24.64 | 23.74 |
| 3.00 | 29.89 | 28.75 | 27.68 | 27.17 | 26.68 | 26.20 | 25.74 | 24.86 | 24.01 |
| 3.25 | 29.83 | 28.72 | 27.70 | 27.22 | 26.75 | 26.31 | 25.87 | 25.04 | 24.24 |
| 3.50 | 29.77 | 28.70 | 27.72 | 27.26 | 26.82 | 26.39 | 25.99 | 25.20 | 24.45 |
| 4.00 | 29.63 | 28.63 | 27.73 | 27.31 | 26.91 | 26.53 | 26.16 | 25.46 | 24.79 |

Notes. K represents the strike as a percentage of the asset price. T denotes the time to maturity, expressed in years, whereas implied volatilities are expressed in percentage. The parameters used in generating this table are the ones of table 1 together with $r = 0.015$ and $q_1 = 0.03$.

Note that the specification of table 1 assumes that $M_{12} = M_{21} = 0$, $Q_{12} = Q_{21}$ and $M_{11} = M_{22}$. Given these assumptions and taking into account

equations (3) and (4) we have the following long-term covariance matrix:

$$\bar{X} = -\frac{\beta}{2M_{11}} \begin{pmatrix} Q_{11}^2 + Q_{12}^2 & (Q_{11} + Q_{22})Q_{12} \\ (Q_{11} + Q_{22})Q_{12} & Q_{22}^2 + Q_{12}^2 \end{pmatrix} \quad (14)$$

Table 5 provides the initial correlation levels, as well as the long-term correlations corresponding to the parametric specification of table 1. In this case, all the initial correlations are higher, in absolute value, than their long-term counterparts.

Table 4: Implied volatility surface for asset 2 generated by the bivariate Wishart volatility model.

| T/K | 70% | 80% | 90% | 95% | 100% | 105% | 110% | 120% | 130% |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.50 | 31.75 | 28.37 | 24.98 | 23.28 | 21.54 | 19.84 | 18.17 | 15.35 | 13.29 |
| 0.75 | 30.36 | 27.50 | 24.72 | 23.36 | 22.03 | 20.72 | 19.44 | 17.09 | 15.25 |
| 1.00 | 29.39 | 26.89 | 24.61 | 23.48 | 22.37 | 21.30 | 20.29 | 18.39 | 16.62 |
| 1.25 | 28.72 | 26.53 | 24.46 | 23.52 | 22.63 | 21.78 | 20.95 | 19.33 | 17.74 |
| 1.50 | 28.07 | 26.09 | 24.43 | 23.67 | 22.92 | 22.18 | 21.45 | 20.02 | 18.60 |
| 1.75 | 27.45 | 25.90 | 24.51 | 23.83 | 23.16 | 22.50 | 21.84 | 20.55 | 19.27 |
| 2.00 | 27.14 | 25.88 | 24.59 | 23.97 | 23.35 | 22.74 | 22.14 | 20.97 | 19.81 |
| 2.25 | 27.02 | 25.76 | 24.66 | 24.08 | 23.50 | 22.94 | 22.38 | 21.30 | 20.24 |
| 2.50 | 26.96 | 25.83 | 24.71 | 24.16 | 23.62 | 23.09 | 22.57 | 21.57 | 20.59 |
| 2.75 | 26.91 | 25.82 | 24.74 | 24.22 | 23.71 | 23.22 | 22.74 | 21.81 | 20.90 |
| 3.00 | 26.87 | 25.80 | 24.76 | 24.25 | 23.79 | 23.32 | 22.87 | 22.00 | 21.15 |
| 3.25 | 26.82 | 25.77 | 24.77 | 24.30 | 23.85 | 23.41 | 22.98 | 22.16 | 21.37 |
| 3.50 | 26.76 | 25.74 | 24.78 | 24.33 | 23.90 | 23.48 | 23.08 | 22.31 | 21.56 |
| 4.00 | 26.63 | 25.66 | 24.78 | 24.37 | 23.98 | 23.60 | 23.24 | 22.54 | 21.87 |

Notes. K represents the strike as a percentage of the asset price. T denotes the time to maturity, expressed in years, whereas implied volatilities are expressed in percentage. The parameters used in generating this table are the ones of table 1 together with $r = 0.015$ and $q_2 = 0.03$.

Note that since the sensitivity of the outperformance option with respect to the instantaneous correlation of assets returns is negative, intuitively, a lower long-term correlation should imply a higher premium associated with the outperformance option, above all for long-term options. Pan et al. (2001)

performed an empirical analysis of the term structure of correlation using data from European stock markets. They concluded that correlation tends to increase with the return horizon. Nevertheless, they found several cases showing that correlations decline when the investment horizon increases.

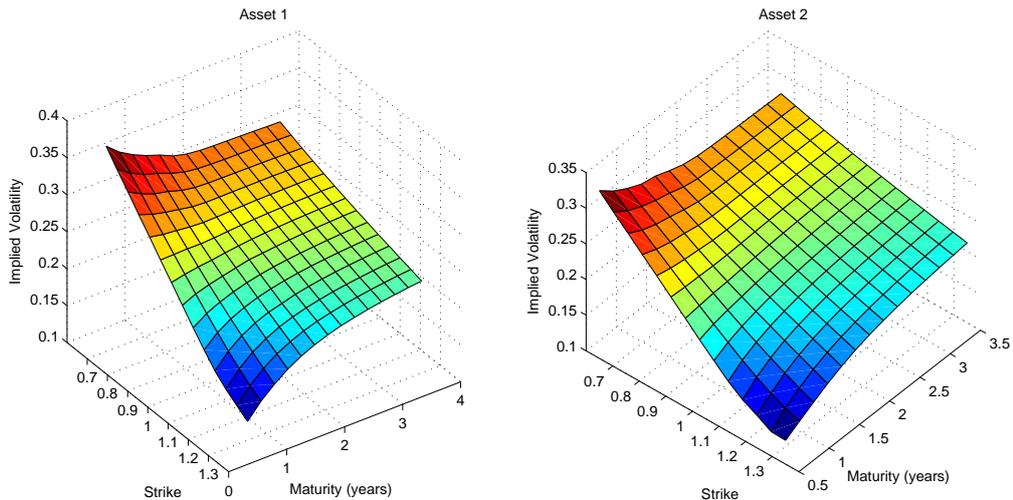


Figure 3: Implied volatility surfaces generated by the bivariate Wishart volatility model. Strike prices are expressed as a percentage of the asset price, whereas maturities are expressed in years. The surfaces are generated using the parameters of table 1 together with $q_i = 0.03$, for $i = 1, 2$ and $r = 0.015$.

Table 5: Correlations associated with the specification of table 1 corresponding to the bivariate Wishart volatility model.

| Initial levels | | Long-term levels | |
|--------------------------|---------|-----------------------------|---------|
| $\rho_{(0)V_{11}V_{22}}$ | 0.7253 | $\bar{\rho}_{V_{11}V_{22}}$ | 0.6696 |
| $\rho_{(0)S_1S_2}$ | 0.7708 | $\bar{\rho}_{S_1S_2}$ | 0.5825 |
| $\rho_{(0)S_1V_{22}}$ | -0.6489 | $\bar{\rho}_{S_1V_{22}}$ | -0.4904 |
| $\rho_{(0)S_2V_{11}}$ | -0.6608 | $\bar{\rho}_{S_2V_{11}}$ | -0.4993 |

Notes. The long-term levels are calculated using \bar{X} instead of X_0 .

Taking into account the long-term covariance matrix of equation (14), the long-term correlation corresponding to the assets returns depends exclusively

on the elements of the matrix Q , which characterizes the randomness of the state variable X , and takes the following form:

$$\bar{\rho}_{S_1 S_2} = \frac{(Q_{11} + Q_{22}) Q_{12}}{\sqrt{(Q_{11}^2 + Q_{12}^2)(Q_{22}^2 + Q_{12}^2)}}$$

Therefore, different combinations of parameters can lead to different patterns corresponding to the term structure of correlation. In particular, if for the parametric specification of table 1 we consider $Q_{12} = Q_{21} = 0.17$, we have the following correlation levels:

$$\begin{aligned} \rho_{(0)V_{11}V_{22}} &= 0.7911; & \bar{\rho}_{V_{11}V_{22}} &= 0.7842 \\ \rho_{(0)S_1S_2} &= 0.7708; & \bar{\rho}_{S_1S_2} &= 0.7539 \\ \rho_{(0)S_1V_{22}} &= -0.6864; & \bar{\rho}_{S_1V_{22}} &= -0.6714 \\ \rho_{(0)S_2V_{11}} &= -0.6925; & \bar{\rho}_{S_2V_{11}} &= -0.6774 \end{aligned}$$

In this case, the initial level of correlation between assets returns is the same as in table 1 and the long-term correlation is very close to the initial correlation level. In this sense, the term structure of correlation is almost flat. I denote the specification of table 1 with $Q_{12} = 0.12$ the negative term structure (NTS) specification and the specification with $Q_{12} = 0.17$ the flat term structure (FTS) specification. To analyze the importance of the term structure of correlation in the pricing of the European outperformance option, table 6 compares the prices obtained using both specifications for different maturities and moneyness⁸. Regarding the NTS specification, the table provides the prices associated with the European outperformance option, whereas for the FTS specification the table shows the percentage difference⁹.

As expected, the prices associated with the NTS specification are higher than the prices corresponding to the FTS specification. For the at-the-money,

⁸The moneyness is defined as $m_0 = S_{10} - S_{20}$. The price of asset 2 is set equal to 1 and the price of asset 1 lies within the interval $[0.8, 1.2]$.

⁹This percentage difference is calculated as $FTS_price/NTS_price - 1$, where FTS_price is the price obtained under the FTS specification and NTS_price is the price corresponding to the outperformance option under the NTS specification.

as well as for the in-the-money options, the percentage difference is increasing with the maturity. Conversely, for the out-of-the-money options this difference decreases with the maturity. Importantly, the results of table 6 show that the term structure of correlation has a relevant effect on the prices of the outperformance option. Therefore, the consideration of a stochastic correlation framework that correctly accounts for this effect is crucial for the correct valuation of this kind of options that exhibit sensitivity to correlation.

Table 6: Prices corresponding to European outperformance options associated with different term structures of correlation.

| m_0 | $T = 0.5$ | | $T = 1$ | | $T = 2$ | | $T = 3$ | |
|-------|-----------|--------|---------|--------|---------|--------|---------|--------|
| | NTS | FTS | NTS | FTS | NTS | FTS | NTS | FTS |
| -20 | 0.42 | -51.13 | 1.62 | -44.52 | 4.13 | -37.47 | 6.22 | -33.82 |
| -10 | 1.80 | -32.46 | 4.14 | -29.59 | 7.64 | -26.73 | 10.14 | -25.28 |
| 0 | 5.62 | -15.05 | 8.57 | -17.21 | 12.45 | -18.31 | 15.08 | -18.59 |
| 10 | 12.29 | -5.51 | 14.83 | -9.23 | 18.42 | -12.25 | 20.90 | -13.55 |
| 20 | 20.76 | -1.97 | 22.46 | -4.82 | 25.33 | -8.12 | 27.45 | -9.86 |

Notes. The moneyness is defined as $m_0 = S_{10} - S_{20}$, The price of asset 2 is set equal to 1 and the price of asset 1 lies within the interval $[0.8, 1.2]$. T denotes the time to maturity expressed in years. Regarding the NTS specification, the table shows the prices corresponding to the European outperformance option, whereas for the FTS specification the table provides the percentage difference calculated as $FTS_price/NTS_price - 1$, where FTS_price is the price obtained under the FTS specification and NTS_price is the price corresponding to the outperformance option under the NTS specification. The moneyness, the option prices and the percentage difference are expressed in percentage.

4 The importance of stochastic correlation

This section provides a numerical illustration which shows the advantages of using the Wishart specification with stochastic correlation with respect to other multi-asset models with constant correlations widely used by financial institutions to price multi-asset options. In particular, I consider a multi-

asset local volatility model and a multi-asset version of the Heston (1993) model. The main reason for this choice is that this model is one of the most popular models within the class of stochastic volatility models. As a starting point I also consider the analytical formula of equation (12) to price outperformance options under the MBS model.

In this section I follow the methodology introduced by Hull and Suo (2002) to measure the model risk¹⁰ embedded in the pricing of exotic options. Note that the key point of model risk is that different models can yield the same price for plain vanilla options but, at the same time, very different prices for exotic options depending on their assumptions corresponding to the evolution of the underlying assets prices, their volatilities and their correlations. In this sense, I assume that market prices are governed by the specification of table 1 corresponding to the Wishart stochastic volatility model. I then calibrate the parameters associated with the other models to the market prices of vanilla options and I use these parameters to price European outperformance options. To this end, I now present the main features of the models considered.

4.1 Models specifications

4.1.1 The local volatility model

The local volatility model was introduced by Dupire (1994), Derman and Kani (1994) and Rubinstein (1994). Under this model the assets prices evolution is governed by the following stochastic differential equations, under the risk-neutral probability measure Θ :

$$\frac{dS_{it}}{S_{it}} = (r - q_i) dt + \sigma_{Li}(t, S_{it}) dW_{it}^{\Theta} \quad i = 1, 2 \quad (15)$$

¹⁰Model risk is the risk that arises from the utilization of an inadequate model.

where W_{it}^Θ is a Wiener process under Θ and the instantaneous volatility, denoted local volatility, $\sigma_{Li}(t, S_{it})$ is a function of time and asset price. I assume that the constant instantaneous correlation corresponding to the Wiener processes coincides with the initial correlation between assets returns associated with the bivariate Wishart volatility model:

$$dW_{1t}^\Theta dW_{2t}^\Theta = \rho_{(0)S_1 S_2} dt$$

The local volatility model is able to account for the volatility skew observed in the market prices of vanilla options corresponding to individual underlying assets. But, since the local volatility is a deterministic function of time and the spot price, the local volatility model does not properly accounts for the existence of volatility in the volatility.

I calibrate the local volatility model to provide a close fit as possible to the individual implied volatility surfaces generated by the Wishart specification. To this end, I use the approach introduced by Marabel (2010) to calculate the local volatility. This methodology consists of smoothing the implied volatility through a flexible parametric function, which is consistent with the no-arbitrage conditions developed by Lee (2004) for the asymptotic behavior of the implied volatility at extreme strikes. The local volatility function is then calculated analytically. This approach allows us obtaining smooth and stable local volatility surfaces while capturing the prices of vanilla options quite accurately.

4.1.2 Multivariate Heston stochastic volatility framework

I consider the following risk-neutral dynamics corresponding to the evolution of the assets prices and their instantaneous variances v_{it} , for $i = 1, 2$, under

the probability measure Θ :

$$\begin{bmatrix} \frac{dS_{1t}}{S_{1t}} \\ \frac{dS_{2t}}{S_{2t}} \\ dv_{1t} \\ dv_{2t} \end{bmatrix} = \begin{bmatrix} (r - q_1) \\ (r - q_2) \\ \kappa_1 (\theta_1 - v_{1t}) \\ \kappa_2 (\theta_2 - v_{2t}) \end{bmatrix} dt + \begin{bmatrix} \sqrt{v_{1t}} dW_{S_1,t}^\Theta \\ \sqrt{v_{2t}} dW_{S_2,t}^\Theta \\ \zeta_1 \sqrt{v_{1t}} dW_{v_1,t}^\Theta \\ \zeta_2 \sqrt{v_{2t}} dW_{v_2,t}^\Theta \end{bmatrix} \quad (16)$$

where the parameter θ_i ($i = 1, 2$) represents the long-term mean corresponding to the instantaneous variance of the underlying asset i , κ_i denotes the speed of mean reversion and, ζ_i represents the volatility of the variance.

Regarding the assets returns correlations, the inter-asset-variance correlations, the inter-variance correlations and the cross-asset-variance correlations, I assume that these parameters are equal to the initial correlation levels associated with the Wishart specification. Therefore, the correlation matrix associated with the Wiener processes corresponding to the multi-asset version of the Heston (1993) model is given by:

$$\Upsilon = \begin{pmatrix} 1 & \rho^{(0)}_{S_1 S_2} & \rho_{S_1 V_{11}} & \rho^{(0)}_{S_1 V_{22}} \\ \rho^{(0)}_{S_1 S_2} & 1 & \rho^{(0)}_{S_2 V_{11}} & \rho_{S_2 V_{22}} \\ \rho_{S_1 V_{11}} & \rho^{(0)}_{S_2 V_{11}} & 1 & \rho^{(0)}_{V_{11} V_{22}} \\ \rho^{(0)}_{S_1 V_{22}} & \rho_{S_2 V_{22}} & \rho^{(0)}_{V_{11} V_{22}} & 1 \end{pmatrix}$$

On the other hand, regarding the calibration of the other parameters corresponding to this multivariate version of the Heston (1993) stochastic volatility model, we can use the well known pricing formula for European options¹¹ to calibrate the parameters θ_i , κ_i , ζ_i , and v_{i0} (the initial level corresponding to the instantaneous variance) to the European option prices generated by the Wishart specification for each underlying asset.

4.2 Calibration results

For the local volatility model and for the Heston (1993) model the parameters are chosen to provide as close a fit as possible to the implied volatility surfaces

¹¹I use the fast Fourier transform proposed by Carr and Madan (1999) to improve the accuracy of the results.

associated with asset 1 and asset 2 under the parametric specification of table 1 corresponding to the multivariate Wishart model. For all the specifications I assume that $r = 0.015$ and $q_i = 0.03$ for $i = 1, 2$. Regarding the Heston (1993) model, the estimated parameters corresponding to asset 1 are $\theta_1 = 0.0906$, $\kappa_1 = 2.0300$, $\zeta_1 = 0.5120$, and $v_{10} = 0.0529$. On the other hand, the model parameters associated with asset 2 are $\theta_2 = 0.0717$, $\kappa_2 = 2.0300$, $\zeta_2 = 0.4518$, and $v_{20} = 0.0426$.

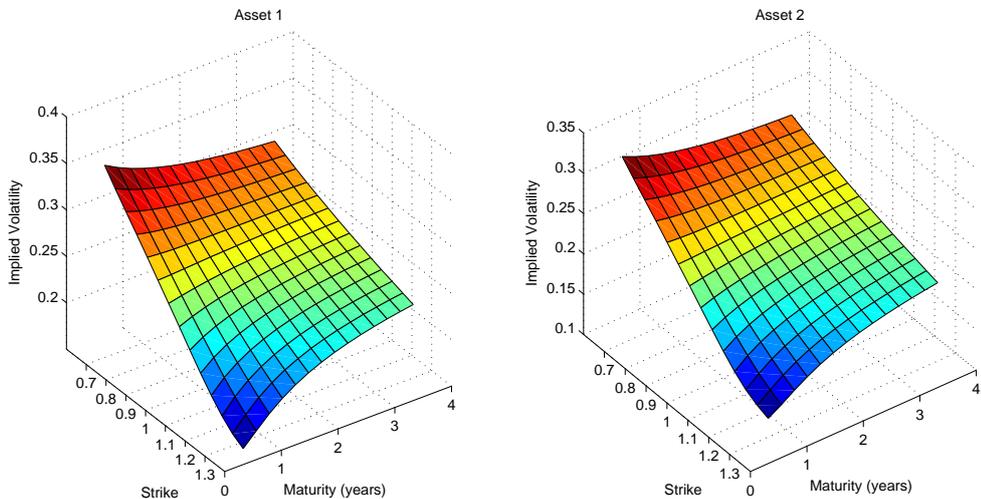


Figure 4: Calibrated implied volatility surfaces corresponding to the Heston model. Strike prices are expressed as a percentage of the asset price, whereas maturities are expressed in years. The surfaces are calibrated to the specification of table 1 corresponding to the Wishart model using the following parameters values $q_i = 0.03$, for $i = 1, 2$ and $r = 0.015$.

Panel A of table 7 reports the mean absolute error (MAE) corresponding to the difference between the implied volatility surface generated by the Wishart specification of table 1 for asset 1 and the implied volatility calibrated using the local volatility model and the Heston (1993) model. The table provides information about the total MAE and the MAE associated with at-the-money options. Analogously, panel B shows the estimation er-

rors corresponding to the implied volatility associated with asset 2.

Table 7: Mean absolute error (MAE) associated with the difference between the implied volatilities generated by the Wishart specification of table 1 and the calibrated implied volatilities corresponding to the local volatility model and the Heston model.

| Panel A: asset 1 | | | |
|---------------------------------|------------------|--------|--|
| | Local volatility | Heston | |
| <i>Total MAE</i> | 0.060% | 0.188% | |
| <i>MAE at-the-money options</i> | 0.032% | 0.136% | |
| Panel B: asset 2 | | | |
| | Local volatility | Heston | |
| <i>Total MAE</i> | 0.068% | 0.151% | |
| <i>MAE at-the-money options</i> | 0.037% | 0.045% | |

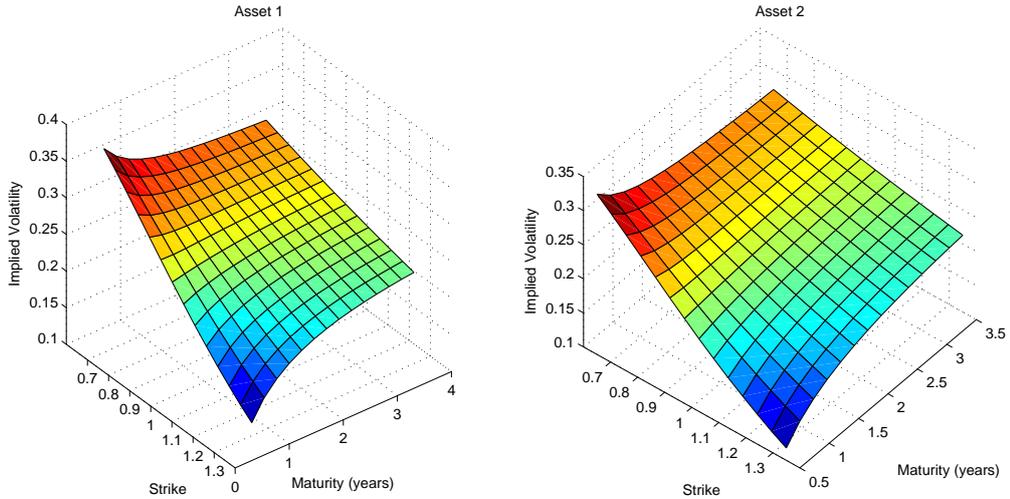


Figure 5: Calibrated implied volatility surfaces corresponding to the local volatility model. Strike prices are expressed as a percentage of the asset price, whereas maturities are expressed in years. The surfaces are calibrated to the specification of table 1 corresponding to the Wishart model using the following parameters values $q_i = 0.03$. for $i = 1, 2$ and $r = 0.015$.

The results show that both models provide an accurately fit to the implied volatility surfaces generated by the Wishart specification, specially the local

volatility model. Figure 4 shows the implied volatility surfaces generated by the calibrated parameters corresponding to the Heston (1993) model for both underlying assets, whereas figure 5 provides the implied volatility surfaces associated with the local volatility model.

4.3 Pricing performance

Once we have calibrated the local volatility model and the Heston (1993) model to the implied volatilities generated by the parametric specification of table 1 associated with the Wishart model, it is possible to compare the pricing performance of both specifications in the valuation of European out-performance options. I also compare these prices with the prices obtained under the assumptions of the MBS model. To price the outperformance options under the multi-asset version of the Heston (1993) model and the local volatility model, I use Monte Carlo simulations with weekly time steps and 100.000 trials, based on the dynamics of equation (16) and (15) respectively. I apply the antithetic variable technique described in Boyle (1977) to reduce the variance of the estimates and, for the Heston (1993) model, I implement a Milstein discretization scheme as described in Gatheral (2006). Regarding the MBS model, I consider the analytical expression of equation (12). In this case, the instantaneous volatilities of each underlying asset are set equal to the at-the-money implied volatility generated by the Wishart specification of table 1 for each maturity. On the other hand, the correlation between assets returns is set equal to the initial correlation level obtained under the Wishart specification:

$$\lambda = \rho_{(0)S_1S_2} = 0.7708$$

Table 8: Prices corresponding to European outperformance options with maturity within one year under different models.

| m_0 | Wishart | MBS | Local | Heston |
|-------|----------------|------------|--------------|---------------|
| -20 | 1.62 | -66.75 | -21.44 | -54.23 |
| -10 | 4.14 | -44.38 | -27.61 | -41.12 |
| 0 | 8.57 | -27.00 | -26.50 | -21.48 |
| 10 | 14.83 | -15.54 | -19.09 | -12.35 |
| 20 | 22.46 | -8.60 | -10.94 | -5.95 |

Notes. *Wishart* represents the prices corresponding to outperformance options under the multi-asset *Wishart* volatility model using equation (13) and the specification of table 1. For the other models the table shows the percentage difference in prices with respect to the *Wishart* model. The prices have been calculated assuming that $r = 0.015$ and $q_i = 0.03$, for $i = 1, 2$. See also notes for table 6.

Table 9: Prices corresponding to European outperformance options with maturity within three years under different models

| m_0 | Wishart | MBS | Local | Heston |
|-------|----------------|------------|--------------|---------------|
| -20 | 6.22 | -48.21 | -31.03 | -44.86 |
| -10 | 10.14 | -36.92 | -31.38 | -33.65 |
| 0 | 15.08 | -27.81 | -28.90 | -25.52 |
| 10 | 20.90 | -20.72 | -23.82 | -18.17 |
| 20 | 27.45 | -15.34 | -18.51 | -12.79 |

Notes. *Wishart* represents the prices corresponding to outperformance options under the multi-asset *Wishart* volatility model using equation (13) and the specification of table 1. For the other models the table shows the percentage difference in prices with respect to the *Wishart* model. The prices have been calculated assuming that $r = 0.015$ and $q_i = 0.03$, for $i = 1, 2$. See also notes for table 6.

Table 8 reports the prices of European outperformance options with maturity equal to one year for different moneyness calculated using the multivariate *Wishart* model¹² under the specification of table 1, as well as the

¹²Note that the prices corresponding to the *Wishart* model coincide with the ones of table 6 associated with the NTS specification.

percentage difference in prices when the option prices are calculated using the local volatility model, the Heston (1993) model and the MBS model. On the other hand, table 9 shows these results corresponding to outperformance options with maturity within three years.

Note that the MBS model does not take into account the implied volatility skew observed in the market prices of plain vanilla options for each underlying asset. The difference between the prices obtained under this model and under the local volatility model reflects the skew premium. As said previously, the local volatility model does not properly accounts for the existence of volatility in the volatility. In this sense, the difference between the prices obtained under this model and the prices generated by the multi-asset Heston (1993) volatility model reflects the stochastic volatility premium. Note that only the Wishart model considers the existence of stochastic correlation. Hence, the difference between the prices obtained under this specification and under the multi-asset Heston (1993) model accounts for the stochastic correlation premium. The results of table 8 and table 9 show that this effect is quite relevant and it is increasing with the maturity for at-the-money and in-the-money options. For all the models, the major discrepancies with respect to the Wishart specification occurs for the out-of-the-money options. Hence, the consideration of stochastic correlation is specially crucial for this kind of options.

Importantly, tables 8 and 9 show that a model that assumes constant instantaneous correlations is not able to yield the same prices for outperformance options as the multi-asset Wishart volatility model with stochastic correlation. Note that, in addition to presenting more flexibility by allowing the existence of stochastic correlation, the Wishart specification allows us pricing the outperformance of options analytically. This is not possible un-

der the local volatility model and under the multi-asset Heston (1993) model. Therefore, the multi-asset Wishart specification includes the additional advantage of analytical tractability. These results have important implications: the consideration of stochastic correlation is a key element for the correct valuation of multi-asset exotic options with exhibit sensitivity to correlation. Therefore, it can be dangerous to assume constant instantaneous correlations to price multi-asset products such as worst of or best of performing structures.

5 Conclusion

The growth experienced in recent years in both the variety and volume of structured products with embedded multi-asset options, which are traded in the over-the-counter market, implies that banks and other financial institutions have become increasingly exposed to correlation between the underlying assets. The empirical evidence shows that the correlations between financial quantities change over time and are notoriously unstable. Moreover, there is a link between correlation and volatility: correlation between financial assets tends to be high in periods of high market volatility. In particular, in 2008 the crisis in the credit market led to a spike in correlations and volatilities that resulted in large losses for several exotic trading desks corresponding to different financial institutions.

Despite the previous considerations, multi-asset options are usually priced assuming constant instantaneous correlations between assets. In this sense, this article introduces a multi-asset model based on Wishart processes that accounts for stochastic volatility and for stochastic correlations between the underlying assets, as well as between their volatilities. The model accounts for the existence of correlation term structure and allows for a positive link

between the assets returns correlation and their volatility that is consistent with the empirical evidence. In this sense, the model is able to account for the existence of correlation skew.

Another advantage of the model is its analytical tractability: it is possible to use the transform pricing approach for affine processes from Duffie et al. (2000) to derive semi-closed-form solutions for European options on each underlying, as well as for the price of multi-asset exotic options such as the outperformance option. Therefore, it is possible to use these exotic options as instruments to calibrate the parameters of the model to real market data.

Importantly, the article shows that the price of the outperformance option depends crucially on the term structure of the correlation corresponding to the assets returns. Moreover, to motivate the introduction of the Wishart volatility model, I compare the prices obtained under this model and under other models with constant correlations commonly used by financial institutions. In particular, I consider a multi-asset local volatility model and a multi-asset version of the Heston (1993) stochastic volatility model. The empirical results show that the consideration of stochastic correlation is a key element for the correct valuation of multi-asset exotic options with exhibit sensitivity to this variable. Hence, it can be dangerous to assume constant instantaneous correlations to price this kind of products.

A The Fourier transform

Let us consider the security $G_{a,b}(y; Y_0, T)$ defined by equation (10). As said previously, Duffie et al. (2000) show that the Fourier transform of $G_{a,b}(y; Y_t, T - t)$ is given by the following expression:

$$\Psi_{a,b}(z; Y_t, T - t) = \int_{-\infty}^{\infty} e^{izy} dG_{a,b}(y; Y_t, T - t) = E_{\Theta} \left[e^{-r(T-t)} e^{\gamma' Y_T} \right]$$

From the Feynman-Kac theorem we have that:

$$\begin{aligned} \frac{\partial \Psi_{a,b}}{\partial t} &+ \sum_{i=1}^2 \frac{\partial \Psi_{a,b}}{\partial Y_{it}} E_{\Theta} [dY_{it}] + \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial \Psi_{a,b}}{\partial X_{ijt}} E_{\Theta} [dX_{ijt}] \\ &+ \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2 \Psi_{a,b}}{\partial Y_{it} \partial Y_{jt}} Cov_{\Theta} [dY_{it}, dY_{jt}] \\ &+ \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 \frac{\partial^2 \Psi_{a,b}}{\partial X_{ijt} \partial X_{kmt}} Cov_{\Theta} [dX_{ijt}, dX_{kmt}] + \\ &\sum_{i=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 \frac{\partial^2 \Psi_{a,b}}{\partial Y_{it} \partial X_{kmt}} Cov_{\Theta} [dY_{it}, dX_{kmt}] = r \Psi_{a,b} \end{aligned}$$

Substituting the expressions corresponding to the expectations and covariances in the previous equation yields:

$$\begin{aligned} \frac{\partial \Psi_{a,b}}{\partial t} &+ \sum_{i=1}^2 \frac{\partial \Psi_{a,b}}{\partial Y_{it}} \left(r - q_i - \frac{1}{2} X_{iit} \right) + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2 \Psi_{a,b}}{\partial Y_{it} \partial Y_{jt}} X_{ijt} \quad (17) \\ &+ \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial \Psi_{a,b}}{\partial X_{ijt}} \left(\Omega \Omega' + M X_t + X_t M' \right)_{ij} \\ &+ \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 \frac{\partial^2 \Psi_{a,b}}{\partial X_{ijt} \partial X_{kmt}} \left(X_{ikt} Q'_j Q_m + X_{imt} Q'_j Q_k \right) \\ &+ \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 \frac{\partial^2 \Psi_{a,b}}{\partial X_{ijt} \partial X_{kmt}} \left(X_{jkt} Q'_i Q_m + X_{jmt} Q'_i Q_k \right) \\ &+ \sum_{i=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 \frac{\partial^2 \Psi_{a,b}}{\partial Y_{it} \partial X_{kmt}} \left(X_{ikt} \rho' Q_m + X_{imt} \rho' Q_k \right) = r \Psi_{a,b} \end{aligned}$$

As pointed out by Bru (1991), the Laplace transform of Wishart processes is exponentially affine. Therefore, I postulate the following guess solution:

$$\Psi_{a,b}(z; Y_t, T - t) = \Psi_{a,b}(z; Y_t, \tau) = e^{B(\gamma, \tau) + \text{tr}[A(\gamma, \tau) X_t] + \gamma' Y_t}$$

where $\tau = T - t$, $\gamma = a + bzi$, $i^2 = -1$ and where $A(\gamma, \tau)$ is a 2×2 matrix.

The boundary conditions associated with the previous solution are:

$$A(\gamma, 0) = 0$$

$$B(\gamma, 0) = 0$$

Substituting the guess solution in equation (17) and dividing by $\Psi_{a,b}$ yields:

$$\begin{aligned} & -\frac{\partial B}{\partial \tau} - \text{tr} \left(\frac{\partial A}{\partial \tau} X_t \right) + \sum_{i=1}^2 \gamma_i \left(r - q_i - \frac{1}{2} X_{iit} \right) \\ & + \sum_{i=1}^2 \sum_{j=1}^2 A_{ji} \left(\Omega \Omega' + M X_t + X_t M' \right)_{ij} + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \gamma_i \gamma_j X_{ijt} \\ & + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 A_{ji} A_{mk} \left(X_{ikt} Q'_j Q_m + X_{imt} Q'_j Q_k \right) \\ & + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 A_{ji} A_{mk} \left(X_{jkt} Q'_i Q_m + X_{jmt} Q'_i Q_k \right) \\ & + \sum_{i=1}^2 \sum_{k=1}^2 \sum_{m=1}^2 \gamma_i A_{mk} \left(X_{ikt} \rho' Q_m + X_{imt} \rho' Q_k \right) - r = 0 \end{aligned}$$

Taking into account that X_t is symmetric and assuming that $A(\gamma, \tau)$ is symmetric as well, we can express the previous equation as follows:

$$\begin{aligned} & -\frac{\partial B}{\partial \tau} - \text{tr} \left[\frac{\partial A}{\partial \tau} X_t \right] + \sum_{i=1}^2 \gamma_i (r - q_i) - \frac{1}{2} \text{tr} [\text{diag}(\gamma) X_t] \\ & + \text{tr} [\Omega \Omega' A] + \text{tr} [A M X_t] + \text{tr} [M' A X_t] + \frac{1}{2} \text{tr} [\gamma \gamma' X_t] \\ & + 2 \text{tr} [A Q' Q A X_t] + \text{tr} [\gamma \rho' Q A X_t] + \text{tr} [A Q' \rho \gamma' X_t] - r = 0 \end{aligned}$$

This equation has to hold for all values of X . Therefore, we have:

$$\begin{aligned} \frac{\partial B}{\partial \tau} & = d_0(\gamma) + \text{tr} [\Omega \Omega' A] \\ \frac{\partial A}{\partial \tau} & = A [M + Q' \rho \gamma'] + [M' + \gamma \rho' Q] A + 2 A Q' Q A + d_1(\gamma) \end{aligned}$$

where:

$$d_0(\gamma) = \sum_{i=1}^2 \gamma_i (r - q_i) - r$$

$$d_1(\gamma) = \frac{1}{2} [\gamma\gamma' - \text{diag}(\gamma)]$$

The solution to the previous equations is given by:

$$A(\gamma, \tau) = C_{22}^{-1}(\gamma, \tau) C_{21}(\gamma, \tau)$$

with:

$$\begin{pmatrix} C_{11}(\gamma, \tau) & C_{12}(\gamma, \tau) \\ C_{21}(\gamma, \tau) & C_{22}(\gamma, \tau) \end{pmatrix} = \exp \left[\begin{pmatrix} M + Q'\rho\gamma' & -2Q'Q \\ d_1(\gamma) & -M' - \gamma\rho'Q \end{pmatrix} \tau \right]$$

$$B(\gamma, \tau) = d_0(\gamma)\tau - \frac{\beta}{2} \text{tr} [\ln [C_{22}(\gamma, \tau)] + \tau (M + Q'\rho\gamma')]$$

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