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Dynamic balance sheet model with liquidity risk

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Abstract

Theoretically optimal responses of banks to various liquidity and solvency shocks are modelled. The proposed framework is based on a risk-adjusted return portfolio choice in multiple periods subject to the default risk related either to liquidity or solvency problems. Performance of the model and sensitivity of optimal balance sheet structures to some key parameters of the model are illustrated in a specific calibrated setup. The results of the simulations shed light on the effectiveness of the liquidity and solvency regulation. The flexible implementation of the model and its semi-analytical solvability allows for various easy applications of the framework for the macro-prudential policy analysis.

Keywords: asset structure, optimal portfolio, banking
JEL Classification: G11, G21, C61

1 Non-technical summary

Banks' balance sheets change their composition due to market conditions implied by a generic growth of credit and debt in the economy and valuation changes, but also due to bank's strategic actions. The strategic driver is particularly interesting and challenging from the modeling perspective since it combines the changes in risk and trends in the economy, the impact of the regulatory regime and the asset and liability management (ALM). These strategic responses to the changing market and regulatory parameters have to be well understood in order to gauge the influence and effectiveness that macro-prudential policies can have on banking system.

Banks optimise their asset and liability structure and although the goals are multi-dimensional the main theme of risk-adjusted return maximisation prevails in their objectives. Banks need to stay profitable given an acceptable level of risk. The profitability and adequate risk-taking increase the capital base of banks and their resilience to the shocks and have a positive signaling effect for investors buying shares or debt issued. Moreover, the regulatory regime imposes on banks the need to retain sufficient amount of capital to withstand the shocks and enough liquid assets to meet their obligations in the majority of plausible scenarios that can materialise in a given horizon. The regulation defines rigid limits to the expansion of the balance sheet and concentration of exposures relative to capital base and liquidity buffers. All these conditions make the managing of banks' balance sheets a challenging process which in consequence is also challenging from the modeling perspective. This complications notwithstanding, practitioners and the research community are seeking for the right framework to correctly model the sensitivity of the balance sheet structure to the changing market conditions.

We propose a multi-period model of a bank maximising its risk-adjusted return on capital given liquidity and solvency constraints. Solvency and liquidity limits are modelled in a 'worst case' manner. It means that only those strategies of banks are admissible that guarantee with a very high probability that the bank remains solvent and liquid. This approach to the risk measurement and limits is reflected in the existent regulatory regime by the Value-at-Risk (VaR) concept, both in the solvency context (Basel II VaR capital constraint or measurement of banks' economic capital) and in the liquidity context (Liquidity-at-Risk internal models of banks' cash-flow distributions, see Matz and Neu (2006)). The proposed model is an extension of the classical stochastic programming tools *à la* Kusy and Ziemba (1986) which were developed to support the asset and liability management process and incorporates the tail risk measures of liquidity and solvency.

The framework that we propose captures the main features of the ALM decision making process under regulatory and internal risk constraints and is computationally tractable and easy to solve and simulate. The model is semi-analytical, i.e. some parts of it have closed form solutions that improve the speed of the Monte-Carlo simulations which are necessary to solve the optimisation program of a bank (using the standard dynamic programming techniques). Therefore, it is applicable in the stress testing context to endogenize bank's reactions to adverse economic and financial conditions to understand re-balancing of banks' asset portfolios following in particular credit risk or funding risk shocks. Moreover, its tractability can be helpful in improving the treatment of banks' decision problems in the DSGE framework.

The simulations performed in a stylised setup of the parameters, yet realistic and reflecting the most important trade-offs that bank is confronted with (e.g. return vs risk or profitability vs solvency and liquidity conditions), provide with the insight into the relationship between the optimal investment choice and the changes in the credit, market and funding risk parameters and in the risk constraints. The results suggest some very non-linear effects that changes in return and volatility parameters can produce for the bank lending behaviour. These effects are related to the changes in relative risk-adjusted return characteristics of the different balance sheet categories, the correlation structure and the constraints binding for some threshold values of the parameters.

The results also give some insight into the policy of the liquidity and solvency regulation. The regulation of the banks' risk taking at the same time in the liquidity and solvency dimension may not be effective. More specifically, an improvement of solvency conditions may induce banks to expose themselves to more liquidity risk. This trade-off revealed in our model supports a general statement about regulation that potential consequences of regulatory policies have to be assessed and weighted against each other with caution.

2 Introduction

The aim of the study is to propose an analytical framework to analyse banks' theoretically optimal responses to changing liquidity and solvency conditions. Liquidity is understood either as a potential to liquidate assets at their book value or to roll-over the maturing funding sources. Notably, the recent financial crises showed that the liquidity shocks can have significant impact on banks' behaviours. Funding risk is actively managed by banks which means that banks try to ensure that they have enough liquid sources to meet the obligations, funding outflows or collateral downgrades. The active management determines the funding mix and it determines the volume of the liquid assets that are used as a counterbalancing capacity against funding outflow and that creates a potential for credit supply. Therefore, it is one of the most important factors of the investment strategy; banks have to remain both liquid and profitable. The solvency conditions have equally important role in constraining the investment strategies of banks. Banks need to keep sufficient capital to withstand the potential losses that they may incur. However, we emphasise the liquidity dimension of our optimisation model since the solvency risk is a more standard component of the micro-founded models of the bank behaviours.[1]

The fact that banks' balance sheets tend to change dynamically may be particularly problematic for financial analysts and banking regulators. A usual static balance sheet assumption (e.g. taken for the stress testing exercises) may be valid only in some very special cases of shocks of the low magnitude and in relatively short periods (in cases it is reasonable to assume that the adjustment may need time for a preparation and coordination of actions by the management of a bank). The dynamic balance sheet feature of stress testing exercises was explained by Henry et al. (2013) and RTF (2015).

There are at least four policy relevant aspects of the relationship between investment strategies and funding conditions.

1. *Macro-prudential policy analysis.* The macro-prudential policies try to create incentives for financial system to smooth the processes of the real sector, in particular to provide enough bank credit for the corporate sector and households. Efficient policy tools have to take into account the complicated dynamics of banks' balance sheets. The proposed framework allows for analysing the possible reaction functions of banks to funding shocks, solvency constraints, credit risk impacting risk-return characteristics of the loan portfolios and the regulatory parameters (e.g. additional capital and liquidity buffers). The macro-prudential policies that take into account banks' optimisation of their balance sheets can replace the usually applied rules of thumb.

2. *Dynamic solvency conditions.* Although a bank may be solvent from an accounting perspective (possessing enough capital at a given point in time to cover expected, and even part of unexpected losses) it may not run any sustainable business. Funding sources may be volatile enough and difficult to roll-over carrying a substantial risk of fire-sales of assets covering an outflow of funding. The proposed model can help to detect balance sheet structures that may lead to a substantial increase of insolvency risk in the future irrespective of currently applied strategies.

3. *Credit supply conditions.* A bank may be forced to increase the share of the liquid assets in its balance sheet to protect against volatility of funding sources. A preventive tactic to increase liquidity buffer may have a negative externality of reduced potential to lend to the economy.

4. *Effectiveness of liquidity management.* Risk based capital regulation has a strong and well-established position in the traditional banking regulation. It is less so in case of liquidity rules imposed on banks. In Basel II they had a form of a code of good practice rather than liquidity indicators and benchmarks. Such an approach was justified by a belief that setting liquidity limits based on very aggregate supervisory data cannot be an effective tool to control

[1] See for instance Danielsson et al. (2002); Hilberink and Rogers (2002); Cuoco and Liu (2006).

banks' liquidity conditions. Contrary to solvency measures, aggregation for liquidity purposes gives a very imprecise picture. Therefore Basel II related liquidity standards focussed on promoting investment in internal liquidity management systems satisfying some high level principles. Basel III and CRDIV departed from that approach and imposed minimal liquidity ratios (short-term Liquidity Coverage Ratio (LCR) and longer-term Net Stable Funding Ratio (NSFR)). This regulation addresses the causes of the crisis that erupted in 2007 and tries to capture these characteristics of balance sheet items that are important from a liquidity perspective (e.g. stability of funding sources, generally expected haircuts on certain asset classes or operational relationships with customers). This notwithstanding, the LCR and NSFR liquidity rules also simplify the liquidity risk measurement and cannot replace a fully-fledged statistical and behavioural cash management, collateral management and sustainability of funding sources. We apply in our framework the Value-at-Risk based liquidity measures.

To model the rational choices of banks about the structuring of their balance sheets some portfolio optimisation techniques can be used. The deliberate actions taken by banks that change the composition of the balance sheet have their rational economic goals. The complexity of the balance sheet management is related to the fact that it is a multi-criteria problem with goals changing in time depending on the liquidity and solvency outlook. Banks, as all other firms try maximise their profits but also have to build adequate buffers against possible fluctuations of their funding, especially given the high leverage of the banks' business model. Nonetheless, the optimisation tools, especially adapted to the portfolio choice problems in financial mathematics, provide a rich, theoretically well-founded toolkit to describe the process of the risk-adjusted profit maximisation in which banks are involved on a regular basis in their risk management and ALM activities.

The optimisation approach to the banks' asset and liability management dates back to Kusy and Ziemba (1986).[2] The whole strand of research uses the stochastic programming techniques from operations research (Consigli and Dempster, 1998; Klaassen, 1998; Robert and Weissensteiner, 2011), which in practice are well-suited for risk management problems operating with granular portfolios and requiring many decision variables. Bank balance sheet problems can be tackled with approaches of optimal portfolio choice with transaction costs (see Davis and Norman (1990); Hilberink and Rogers (2002)). Transaction costs introduce stickiness in the rebalancing of the portfolio structure by modeling of costly liquidation of a position. This approach is particularly appealing for banks'balance sheets containing illiquid positions like loans or off-the-counter corporate bonds. However, the theory of portfolio choice with transaction costs easily produces computationally untractable problems. They work in a very simplified set-up not allowing for a usual heterogeneity of balance sheet categories of banks' books managed under various stringent capital and liquidity rules. Another interesting, potentially applicable strand of research is related to the real option theory of illiquid investment opportunities. In liability-driven asset management it has already been employed by Ang et al. (2013). Some recent studies apply the robust optimisation tools (Gülpinar and Pachamanova, 2013) that reduce variability of portfolio structures and are computationally tractable.

Banks' balance sheet optimal choice became part of many applications of the optimal portfolio choice theory to the micro-foundation of banks behaviour and regulation. There are two broad groups of approaches in this strand of literature. First, the DSGE models try to characterise the equilibrium behaviours of different stylised types of market players in the economy including banks with the intermediation function (Darracq Paries et al., 2011; Aoki and Sudo, 2013; Clerc et al., 2014). In this approach it is possible to endogenize the parameters of portfolio choice model taking into account the aggregate effects of the decisions of banks and other market players. In contrast, we take the parameters as given but are able to operate in a tractable way with a richer structure of the optimisation problem. Second, Zhu (2008); Acharya (2003) examine the impact of regulation on banks' investment and funding decisions to inform the optimal regulatory regime and assess the credit supply conditions. The models in this group are similar in the setup to the one we propose

[2]In fact, the quantitative approaches to asset-liability management are more advanced in insurance context, see Iyengar (2010). Some papers present very applicable solutions, e.g. Hilli et al. (2007).

but we focus on the shock transmission channels and nonlinearities related to regulatory as well as to the tail risk constraints.

Our model is similar to Birge and Júdice (2013). The main differences of our approach stem from inclusion of risky funding volumes, explicit treatment of fire-sales and liquidity counterbalancing buffers. We introduce the regulatory capital constraint to our model by means of the capital ratio (i.e. the risk-weighted assets over Core Tier 1 capital base). In this respect, our modelling approach is also closely related to the dynamic balance sheet model of Hałaj (2013). However, we introduce also risk sensitive capital and liquidity limits defined by the Value-at-Risk (VaR) of a bank's capital and Liquidity-at-Risk (LaR) of the funding sources. The concept of the VaR of the capital is closely related to the VaR constraint embedded into the regulatory regime of Basel II (and Basel III) inducing banks to structure their balance sheet in such a way that they should remain solvent with a high (predetermined) probability. LaR is similar in philosophy to VaR used for the solvency regulation (Matz and Neu, 2006). It limits bank's exposures to the liquidity risk by allowing for taking only these liquidity positions that with high probability ensure that the bank can meet its obligations in the near future. Alternatively, the liquidity risk limits could be approached via the regulatory LCR constraint (Balasubramanyan and VanHoose, 2013). It is easy to conceptually extend our model to cover the LCR limits but it would be difficult to parameterise this type of a constraint for the very aggregate balance sheet that we operate with.

In the traditional banking literature (Baltensperger, 1980; Boyd and Nicoló, 2005; Pelizzon and Schaefer, 2005) banks are assumed to take investment decisions under the risk neutrality assumption.[3] Risk impacts banks' decisions only via regulatory constraints. We follow an approach from a different strand of literature (Howard and Matheson, 1972; Danielsson et al., 2002; Cuoco and Liu, 2006) where decisions are risk sensitive. Banks are assumed to be risk averse in our setup what follows the capital management practice in banks where RAROC[4] and RARORAC are common indicators for managing accepted levels of the risk exposure and are the standard part of the ALM indicators (Adam, 2008).[5] In fact, Baltensperger (1980) admits that joint modelling of loan or deposit volumes and diversification within these two portfolios can be approached rather by utility maximisation (implying risk sensitive decisions) than expected profit maximisation.

Summing up, the main contribution of our paper to the banking literature is to provide a tool to study endogenous balance sheet structures implied by both funding conditions and investment opportunities. It has a reduced form that improves tractability of a constrained optimisation with VaR-based risk measurement. Therefore, despite covering features of decision marking process under tail risk constraints it is inexpensive to solve. On the potential application side, the framework can be useful in the stress testing context when macro-financial shocks have to be translated into changes of the balance sheet structure. Moreover, it allows for the detection of tensions between the demand for credit and the supply side conditions of banks' balance sheets (insufficient or excessively volatile funding, a poor quality of the counterbalancing capacity or the credit risk accumulated in the outstanding loan portfolios) that may hamper the credit expansion. Moreover, its insight into banks' theoretically optimal responses to some capital shocks and the liquidity requirements may be helpful for the macro-prudential policy assessment, whereby changes in the capital requirements, in the liquidity and credit risk parameters of banks' balance sheets can be translated into the credit supply drivers of some general tendencies in credit growth in the economy.[6]

A sensitivity analysis of the optimal strategies to the parameters of the model that we performed shows some interesting patterns in the potential lending behaviours of banks and also some interesting features of the model itself. First, the correlation between the risk drivers is a very important factor that may change the lending behaviour, especially when the correlation in absolute terms

[3] See also Elyasiani et al. (1995); Balasubramanyan and VanHoose (2013).

[4] Risk-Adjusted Return on Capital and Risk-Adjusted Return on Risk-Adjusted Capital.

[5] The literature of the banking theory is not equivocal about the risk averseness of banks. The traditional strand of literature builds on the assumption of risk neutrality (Baltensperger, 1980; Boyd and Nicoló, 2005; Pelizzon and Schaefer, 2005). In contrast, (Howard and Matheson, 1972; Danielsson et al., 2002; Cuoco and Liu, 2006) models banks' decisions as risk sensitive.

[6] E.g. it can help answering a question of the impact of more stringent capital requirements on banks' propensity to lend in given liquidity conditions and profitability and riskiness of loans.

increases (particularly visible in case of the correlation between credit and market risk). It also affects the volatility of the capital level at the optimum. Second, changes in lending volumes are in a non-linear fashion related to the volatility of securities and the correlation of the risks as an outcome of the interplay of risk constraints (non-linear by their virtue), bank's attempts to stay profitable and bank's preferences of the risk-taking. For instance, the theoretically optimal lending volumes rise as the volatility in the securities portfolio increases until the Value-at-Risk constraint starts to be binding and the lending starts to decline following the maximum lending allowed by the capitalisation of the bank. Third, lending is very sensitive to the relative profitability of the securities and loans – more sensitive than to the relative riskiness of the two asset classes. Fourth, a change in some of the parameters of the balance sheet that improve the solvency conditions of a bank may at the same time deteriorate its exposure to the liquidity risk. That trade-off renders an effective regulatory policy difficult to calibrate.

Notably, the model in this reduced form cannot effectively support the the risk management process of a bank. It operates on a very aggregate balance sheet. The computational complexity of the optimisation based approaches to the assets and liability management is still a problem for current risk management support systems to allow for a comprehensive modelling of balance sheet in its usually observed granularity. Although, there are prototypes of models relying on the risk-return optimisation of banks' exposures used for ALM[7], the increasing power of IT systems is needed to gain an efficiency acceptable in Management Information Systems. Moreover, the model treats parameters as exogenous and several of them (e.g. these referring to the credit quality of lending portfolios) may depend on the choice of banks. It may be particular simplifying if banks' individually optimal portfolio choices are aggregated across the financial system. To address the shortcoming, a DSGE framework can be employed and our computationally tractable model offers an alternative to improve the usually very stylised optimisation of banks' balance sheet structures.

In the following sections we present details (also quite technical) of the modelling approach (section 3) and in section 4 we derive a dynamic programming equation for a 2-period model. Section 5 provides some illustrative examples of parameterisations and sensitivity analysis of the optimal lending strategies and corresponding solvency positions.

3 Model

3.1 Modelling approach

This section presents the details of a model of the asset structure choice of a bank.

In general, we work in a setup of the risk-adjusted return maximisation. The risk is related to uncertain funding sources (the risk of an outflow of deposits), the credit risk in the loan portfolio (outstanding and new volumes treated separately) and the volatile prices of the liquid securities. All these types of risk are accumulated in the balance sheet of a bank that tries to maximise risk adjusted profit given the capital and liquidity constraints. The bank operates in a multi-period (T-period in general, 2-period in the implementation) time frame facing the risk of:

- falling into ill-liquidity when investing too much in loans and exposing itself to a risk of having insufficient liquid funds to meet the obligations at the end of each period;
- falling into insolvency if losses (loan losses and devaluation of securities) erode capital base.

3.2 Risky funding sources

Assumption 1. *There is only one type of funding sources available. It is described by an autoregressive, one factor model. Funding requires to pay a non-random interest rate. Funding risk is correlated, in particular with the value of securities portfolio.*

[7] See Adam (2008).

Bank's funding volumes are assumed to be homogenous (i.e. consisting of only one type of funding sources) and follow a simple autoregressive risky process. We define risk in a probability space with filtration: $\mathcal{P} = (\mathbb{R}, \mathbf{P}, \mathcal{B}(\mathbb{R}), \mathbb{F})$, $\mathbb{F} = (\mathcal{F}_t)_{t=1,2,\ldots}$.[8] Let η be a process on $\mathcal{P}$ describing a shock to the funding sources F, with the following dynamics:

$$F(t+1) = F(t) + \gamma F(t) + \eta(t+1) \tag{1}$$

The initial funding of a bank is denoted $F(0)$.

The funding deposits pay an interest rate r^F, which implies the interest expenses of $C(t) = r^F F(t)$ due at the end of the period $[t-1, t]$.

The change in funding may imply the need to fire sale liquidation of part of the securities portfolio. Fire-sales are triggered by the drop in the stock of funding. The inflow of funding $(F(t) - F(t-1))^+$ is favourable for banks. The outflow $(F(t) - F(t-1))^-$ implies the need to "fire-sale" part of the liquid assets.[9] The loss due to the fire sales is proportional to the liquidated volume which involves a haircut h – to cover an outflow of $(F(t) - F(t-1))^-$, a bank needs to liquidate $(F(t) - F(t-1))^-/(1-h)$.

Parameters γ, r^F and h are deterministic.

3.3 Loans

Assumption 2. *The loan portfolio is homogenous and subject to default risk. Loans pay deterministic interest rate. Loan portfolio is perfectly illiquid, i.e. only the maturing part can be reinvested. The new business has its own default risk characteristics, correlated with the default risk of the outstanding business (as well as with risk factors of securities portfolio and funding).*

Let ρ and ρ^N be some processes on $\mathcal{P}$, taking values from $(-\infty, 1]$, describing a credit quality of the outstanding loan portfolio L and the new origination.

$$L(t+1) = (1-m)L(t)\rho(t+1) + \pi^L(t)\rho^N(t+1) \tag{2}$$

It means that $(1-m)L(t)(1-\rho(t+1))$ units of the outstanding volume of loans defaults between t and $t+1$. The new business volumes are also subject to a default risk: $\pi^L(t)(1-\rho^N(t+1))$ defaults between t and $t+1$. The parameters ρ and ρ^N can be interpreted as loss rates. In practice, they are functions of default distribution (probability of default distribution) and a loss given default (LGD). This observation is important for the application of the model – loss rates are estimated by multiplying random default probability with an average LGD.

Notably, in the current setup of the model, the loan portfolio is homogenous implying the 1-dimensional decision variable π^L. It can be relaxed to a multi-product loan portfolio but one decision variable is enough to capture trade-offs between the return potential, the liquidity risk and the influences of the funding risk on the asset structures. Moreover, the current setup helps to keep the computational cost of the optimisation programm low (defined in section 3.6).

The interest income from loans is measured by the rate payment r multiplied by the end-of-period volume of the loans, i.e.:

$$I(t+1) = rL(t+1)$$

Notably, the interest income of loans is affected by the defaulted volume of loans which is reflected by taking the volume of loans from the end of period $[t, t+1]$ to compute interests earned in that period. An average volume $\frac{L(t+1)+L(t)}{2}$ could be an alternative option, also easy to implement, but leading to a bit more complex formulas. The maturity profile m is constant (and deterministic) and the loan interest rate r is deterministic as well.

[8] $\mathbb{B}$ is a family of the Borel sets.

[9] We label this event "fire-sales" for convenience but it reflects a likely price/volume impact, even though minor, of selling of any given volume of securities for the liquidity purposes. This minor impact can simply be modelled by relatively small h, e.g $h = 0.1\%$.

Credit losses in period $[t, t+1]$, for simplicity directly and fully deducted from capital, are denoted

$$\Delta L(t+1) := L(t+1) - (1-m)L(t) - \pi^L(t)$$

3.4 Securities

Assumption 3. *A homogenous portfolio of securities, the value of which is governed by one risk factor, matures at the end of every period, i.e. it is purchased (completely renewed) at t and is maturing at $t+1$. No short-selling is allowed.*

The part of the value of the balance sheet that is not invested in the loan portfolio is allocated into the securities portfolio. The total reinvestment potential is equivalent to the sum of the maturing loans, the value of securities, the change in funding and the P&L impact of the fire-sales. Notably, the total reinvestment portfolio is impacted by the change in funding asymmetrically depending on the sign of the change: the change of the reinvestment portfolio related to funding is equal to

$$\Delta F_K(t) = F(t) - F(t-1) - \frac{h}{1-h}(F(t) - F(t-1))^-$$

In case of the funding outflow, the bank "fire-sales" its securities to meet the obligations. The price of securities is risky and driven by a stochastic process ϵ, adapted in $\mathcal{P}$.

Therefore, the law of motion for the value of securities is described by

$$S(t+1) = (mL(t) + S(t) + I(t) - C(t) + \Delta F_K(t) - \pi^L(t))\epsilon(t+1) \quad (3)$$

It is assumed that the valuation process accounts for the interest payments. Therefore the P&L impact of securities portfolio is the valuation change

$$\Delta S(t+1) = (mL(t) + S(t) + I(t) - C(t) + \Delta F_K(t) - \pi^L(t))(\epsilon(t+1) - 1)$$

3.5 Capital

Assumption 4. *Capital is a residual part of the balance sheet. At the end of each period $t > 0$ its level changes according to the accrued net interest income generated in period $[t-1, t]$, to valuation changes of securities portfolio and to fire-sales of securities in case of the outflow of funding.*

The dynamics of capital K is implied by the realised P&L with an assumption of no raising of some new capital. The profits of banks are assumed to consist of the interest income and expenses (related to the loans and deposits), the credit losses, the revaluation of the securities portfolio and the valuation changes related to the fire-sales of the securities implied by a decline of funding deposits. For $t = 0$

$$K(0) = L(0) + S(0) - F(0)$$

and for $t > 0$

$$K(t) = K(t-1) + rL(t) - r^F F(t) + \Delta L(t) + \Delta S(t) - \frac{h}{1-h}(F(t) - F(t-1))^-$$

3.6 Goal

Assumption 5. *Bank maximises the sum of discounted risk-adjusted returns from capital subject to liquidity and capital constraints. At each time t, the liquidity constraint assures that with high probability $1-\alpha^F$, for small α^F, bank is able to meet its obligations at $t+1$. Capital constraint is based on a regulatory concept requiring a bank to hold enough capital for the risk weighted assets (minimum 8% capital ratio is a baseline case) combined with the economic capital concept based on VaR calculations. An illiquid or insolvent bank incurs a penalty cost proportional to its volume of the assets.*

A bank is supposed to maximise the expected return on equity, i.e. the expected value of:

$$R(t+1) = \frac{K(t+1) - K(t)}{K(t)} \tag{4}$$

adjusted by the risk of that return and aggregated across periods.

There are two types of constraints imposed on the investment strategy: related to liquidity (LaR) and solvency position (VaR). Liquidity is understood as the balance sheet composition that allows for paying back due liabilities. We omit the cash flow balance of interest paid by loans and funding since we focus on liquidity shocks related to the fluctuations of deposits. For the liquidity purposes a short period Δ^l is assumed – a holding period – in which the liquidity position cannot be adjusted. The investment strategy should then keep enough liquid resources to cover an outflow of deposits in $1-\alpha^F$ fraction of scenarios, at a given confidence level α^F. The securities in the counterbalancing capacity can be liquidated with a haircut h reflecting a discount that can be expected in case of the liquidation (potentially quite high in a 'fire-sales' mode). Putting formally, the LaR constraint has the form:

$$\mathrm{VaR}_\alpha \left(\mathbf{E}\left[(1-h)S(t+\Delta^l) + \big(F(t+\Delta^l) - F(t)\big)|\mathcal{F}_t\right]\right) \geq 0 \tag{5}$$

It can be interpreted as bank's internal requirement to hold enough liquid securities to cover even the α^F worst outflow of the funding sources. Notably, the stock of liquid assets has to increase if their volatility and correlation with the funding sources increases.

Solvency constraints have two forms. One has a very regulatory nature. For the risk weights ω^L and ω^S, and minimum capital ratio κ (e.g. equal to 8%):

$$\kappa(\omega^L((1-m)L(t)+\bar{\pi}) + \omega^S(((m+r)L(t)+S(t)+\Delta F_K(t)-\bar{\pi}))) \leq K(t) \tag{6}$$

which translates into (assuming $\omega^L > \omega^S$):

$$\bar{\pi} \leq \frac{K(t)/\kappa - \omega^L(1-m)L(t) - \omega^S((m+r)L(t)+S(t)+\Delta F_K(t))}{\omega^L - \omega^S}$$

However, banks manage their investment portfolios taking worst case scenarios of capital position in a given Δ^K period into account.[10] We consider Δ^K period forward distribution of income and require that the capital of a bank covers the losses in $(1-\alpha^K)*100\%$ of cases.

$$K(t) + \mathrm{VaR}_{\alpha^K}(\mathbf{E}[K(t+\Delta^K) - K(t)|\mathcal{F}_t]) > 0 \tag{7}$$

Constraints on the strategies at t do not fully prevent the bank from default defined as failing to meet liquidity and solvency requirements at the end of $[t, t+1]$ period. We construct the liquidity process:

$$\mathrm{Liq}(t+1) = (1-h)S(t+1) + (F(t+1) - F(t)),$$

and the solvency process

$$\mathrm{Solv}(t+1) = K(t+1),$$

which is simply the process of capital evolution. We define a stopping time τ on $(\mathbf{P}, \mathbb{F})$ as

$$\tau = \inf\{t|\mathrm{Solv}(t) < 0 \vee \mathrm{Liq}(t) < 0\} \tag{8}$$

portraying an event of a default. Illiquidity or insolvency implies some additional costs for the bank. The costs are assumed to be proportional to the asset volume $L(\tau)+S(\tau)$, with proportionality coefficient ϕ. They are paid once, at the time of default but only if default occurs before the end of the investment horizon T.[11]

[10] E.g. under the ICAAP process.

[11] A precise estimate of the coefficient ϕ may in general be difficult and depending on the jurisdiction, resolution schemes, investors behaviours and complexity of the financial system (Hardy, 2013). For instance, the Lehman case show the complications of the litigation costs due to uncertainty of who owned what to whom.

We assume that banks optimise the risk-adjusted return on capital, aggregated within the horizon of the optimisation. The goal functional is the following:

$$J(l_0, s_0, f_0, \rho_0, \pi) = \mathbf{E}\sum_{t=1}^{T} \delta^t \Big(R(t) - \beta Var(R(t))\Big) - \delta^\tau \phi\Big(S(\tau) + L(\tau)\Big)\mathbb{I}_{\{\tau<T\}} \tag{9}$$

where $Var(R(t)) = \mathbf{E}[(R(t) - \mathbf{E}R(t))^2|\mathcal{F}_{t-1}]$ is a conditional variance process. The goal functional incorporates a penalty cost of either illiquidity or insolvency event. In other words, the bank pays some litigation costs and has to rebuild its reputation when breaching the solvency thresholds or is incapable of covering the outflow of funds with the liquid assets. Consequently, it may loose some potential revenues in the future.

3.7 Towards tractability

The model does not have a closed form solution and can be cumbersome to solve numerically in its general setup. The risk factors in the model are the main source of computational complexity. To simplify the computation we assume that

– risk factors of securities and funding are normally distributed, i.e. factor ϵ has a normal distribution $\mathcal{N}(\mu_\epsilon, \sigma_\epsilon)$, η is normally distributed with mean μ_η and variance σ^2_η;
– risk factors of loans are log-normally distributed, i.e. for normally distributed Z and Z^N, $\rho = 1 - \exp(Z)$ and $\rho^N = 1 - \exp(Z^N)$

The tractability is obtained with some trade-offs, the most important being the negativity of the processes ρ, ρ^N, S and F, although a proper and reasonable calibration may decrease the probability of such an event to close to zero. Moreover, since the returns on financial assets frequently exhibit fat-tails they should be described for instance by means of the so-called stable distributions (Focardi et al., 2013). Consequently, the normality assumption would have implications for underestimation of the VaR of the portfolios. To address this problem, a smaller probability corresponding to the worst outflow or the worst capital ratio than the one implied by the normal distribution with the given estimated variance can be applied. For instance, if the worst case capital ratio is defined by a probability α^K equal to 1% then the VaR_{α^K} can be approximated by a more stringent 5-σ or 7-σ level instead of 2.33-σ. Under the normality assumption, the constraint 5 is significantly simplified. The conditional expectation yields a normally distributed random variable. Given that the VaR periods of liquidity and solvency constraints are different that the profit cycle, the mean and standard deviation parameters have to be scaled, i.e.

$$\mu_{X,l} = \Delta_l \mu_X \qquad \sigma_{X,l} = \sqrt{\Delta_l}\sigma_X$$
$$\mu_{X,K} = \Delta_K \mu_X \qquad \sigma_{X,K} = \sqrt{\Delta_K}\sigma_X$$

Let us denote

$$\begin{aligned} a_\epsilon(t-1,t) &= (m+r)L(t) + S(t) - r^F F(t) + \Delta F_K(t) \\ b_\rho(t) &= (1-m)L(t) \end{aligned}$$

The mean and variance of the conditional expectation are:

$$\begin{aligned} M(t) &= -(1-h)\bar{\pi}\mu_{\epsilon(t+1),l} + (1-h)a_\epsilon(t-1,t)\mu_{\epsilon(t+1),l} + \gamma F(t) + \mu_{\eta(t+1),l} \\ \Sigma^2(t) &= \big((1-h)(a_\epsilon(t-1,t) - \bar{\pi})\sigma_{\epsilon(t+1),l}\big)^2 + \big(\sigma_{\eta(t+1),l}\big)^2 \\ &\quad +2c_{\epsilon(t+1),\eta(t+1)}(1-h)(a_\epsilon(t-1,t) - \bar{\pi})\sigma_{\epsilon(t+1),l}\sigma_{\eta(t+1),l} \end{aligned}$$

and the liquidity condition requires to solve

$$-\frac{M(t)}{\Sigma(t)} \leq \Phi(\alpha^F)$$

where Φ is an inverse function of the standard normal distribution. For usually small α^F, $\Phi(\alpha^F)$ is negative. It means that for $\bar{\pi}$ such that $M(t) \leq 0$, there is no solution. It implies that

$$\bar{\pi} \leq \frac{(1-h)a_\epsilon \mu_{\epsilon,l} + \gamma F(t) + \mu_{\eta,l}}{(1-h)\mu_{\epsilon,l}} = a_\epsilon + \frac{\gamma F(t) + \mu_{\eta,l}}{(1-h)\mu_{\epsilon,l}} \tag{10}$$

if only $(1-h)\mu_{\epsilon,l} > 0$ which is a reasonable assumption. Therefore, equivalently

$$-\Phi(\alpha^F)^2 \Sigma^2(t) + M(t)^2 \geq 0$$

From the two VaR-based constraints, the one related to liquidity conditions is easier to tackle. The liquidity constraint yields a quadratic inequality. A coefficient at the $\bar{\pi}^2$ term is positive for a 'reasonable' parametrisation of the model. The inequality is solved by $\bar{\pi}$ satisfying (for brevity, we drop time indices)

$$\bar{\pi} \in \left[-\infty, \frac{-B - \sqrt{B^2 - 4AC}}{2A}\right] \cup \left[\frac{-B + \sqrt{B^2 - 4AC}}{2A}, +\infty\right] \tag{11}$$

where

$$\begin{aligned}
A &= -\Phi(\alpha^F)^2 (1-h)^2 \sigma_{\epsilon,l}^2 + ((1-h)\mu_{\epsilon,l})^2 \\
B &= -2(1-h)\mu_{\epsilon,l}\left((1-h)a_\epsilon \mu_{\epsilon,l} + \gamma F(t) + \mu_{\eta,l}\right) \\
&\quad +2\Phi(\alpha^F)^2\left[(1-h)^2 \sigma_{\epsilon,l}^2 a_\epsilon + c_{\epsilon,\eta}(1-h)\sigma_{\epsilon,l}\sigma_{\eta,l}\right] \\
C &= \left((1-h)a_\epsilon \mu_{\epsilon,l} + \gamma F(t) + \mu_{\eta,l}\right)^2 \\
&\quad -\Phi(\alpha^F)^2\left[(1-h)^2 \sigma_{\epsilon,l}^2 a_\epsilon^2 + \sigma_{\eta,l}^2 + 2(1-h)c_{\epsilon,\eta} a_\epsilon \sigma_{\epsilon,l}\sigma_{\eta,l}\right]
\end{aligned} \tag{12}$$

Clearly, the coefficient A is positive if

$$\Phi(\alpha^F)^{-2} > \frac{\sigma_{\epsilon,l}^2}{\mu_{\epsilon,l}^2}.$$

For instance, if $\alpha^F = 1\%$ then $\Phi^{-2} \simeq 0.18$ and for return of securities close to 1 (e.g. $\mu_{\epsilon,l} \in [0.9, 1.1]$) the volatility parameter $\sigma_{\epsilon,l}^2$ has to be smaller than 40%.

At $t = 1$, the bank needs to solve a quadratic maximisation problem, calculation of which also applies to asset portfolio choice at $t = 0$. Let us denote by μ_ρ and μ_{ρ^N} the means of log-normal distributions of default risk which are equal to:

$$\mu_\rho = 1 - \exp\left(\mu_Z + \frac{\sigma_Z^2}{2}\right) \quad \mu_{\rho^N} = 1 - \exp\left(\mu_{Z^N} + \frac{\sigma_{Z^N}^2}{2}\right) \tag{13}$$

and standard deviations

$$\sigma_\rho = \exp\left(\mu_Z + \frac{\sigma_Z^2}{2}\right)\sqrt{\exp(\sigma_Z^2) - 1} \quad \sigma_{\rho^N} = \exp\left(\mu_{Z^N} + \frac{\sigma_{Z^N}^2}{2}\right)\sqrt{\exp(\sigma_{Z^N}^2) - 1} \tag{14}$$

To characterise the goal function, we need to calculate also correlation of ρ and ρ^N, and correlation of log-normal ρ and ρ^N with normal variables ϵ and η. The former can be directly calculated and equals

$$\left(\exp\left(\mu_Z + \mu_{Z^N} + \frac{\sigma_Z^2 + \sigma_{Z^N}^2 + 2c_{Z,Z^N}\sigma_Z\sigma_{Z^N}}{2}\right) - \mu_\rho \mu_{\rho^N}\right) / (\sigma_\rho \sigma_{\rho^Z})$$

The later can be obtained via Stein's lemma yielding eg. for ρ and ϵ

$$c_{\rho,\epsilon} = -\exp\left(\mu_Z + \frac{\sigma_Z^2}{2}\right)(c_{Z,\epsilon}\sigma_\epsilon\sigma_Z) / \sigma_\rho\sigma_\epsilon$$

The impact of fire-sales on capital creates another complication in the calculations. To represent the mean and variance of the return process we need three types of integrals calculated in the appendix A, i.e. for $n \in \{0, 1, 2\}$, $a \in \mathbf{R}$, $B_2 < 0$, $B_1 \in \mathbf{R}$ and $B_0 \in \mathbf{R}$:

$$I^{(n)}(a, B_0, B_1, B_2) = \int_{-\infty}^{a} x^n \exp\left(B_2 x^2 + B_1 x + B_0\right) \mathrm{d}x \tag{15}$$

The conditional return $R^c(t) \triangleq \mathbf{E}[R(t+1)|\mathcal{F}_t]$ has a distribution which is a mixture of normal, truncated normal and log-normal random variables with mean and variance given by

$$\begin{aligned}
\mu_{R^c(t)} &\triangleq \mu(L(t), S(t-1), S(t), F(t-1), F(t), \bar{\pi}) \\
&= \Big((1+r)b_\rho(t)\mu_{\rho(t+1)} - b_\rho(t) + (1+r)\bar{\pi}\mu_{\rho^N} - \bar{\pi} + (a_\epsilon(t-1,t) - \bar{\pi})(\mu_{\epsilon(t+1)} - 1) \\
&\quad -r^F(F(t) + \gamma F(t) + \mu_{\eta(t+1)}) + \frac{h}{(1-h)\sqrt{2\Pi}\sigma_\eta}(\gamma F(t) I_\eta^{(0)} + I_\eta^{(1)})\Big) / K(t)
\end{aligned} \tag{16}$$

where

$$I_\eta^{(n)} = I^{(n)}(-\gamma F(t), -\frac{\mu_\eta^2}{2\sigma_\eta^2}, \frac{\mu_\eta}{\sigma_\eta^2}, -(2\sigma_\eta^2)^{-1})$$

$$\begin{aligned}
\Sigma^2_{R^c(t)} &\triangleq \Sigma^2(L(t), S(t), F(t-1), F(t), \bar{\pi}) \\
&= K(t)^{-2}\Bigg(\Big((1+r)b_\rho(t)\sigma_{\rho(t+1)}\Big)^2 + \Big((1+r)\bar{\pi}\sigma_{\rho^N(t+1)}\Big)^2 \\
&\quad + \Big((a_\epsilon(t-1,t) - \bar{\pi})\sigma_{\epsilon(t+1)}\Big)^2 + \Big(r^F\sigma_{\eta(t+1)}\Big)^2 \\
&\quad + \frac{2h}{(1-h)\sqrt{2\Pi}\sigma_\eta}\Big((\gamma F(t))^2 I_\eta^{(0)} + 2\gamma F(t) I_\eta^{(1)} + I_\eta^{(2)}\Big) \\
&\quad - \left(\frac{2h}{(1-h)\sqrt{2\Pi}\sigma_\eta}(\gamma F(t) I_\eta^{(0)} + I_\eta^{(1)})\right)^2 \\
&\quad + 2c_{\rho(t+1),\rho^N(t+1)}(1+r)^2 b_\rho(t)\bar{\pi}\sigma_{\rho(t+1)}\sigma_{\rho^N(t+1)} \\
&\quad + 2c_{\epsilon(t+1),\rho^N(t+1)}(1+r)(a_\epsilon(t-1,t) - \bar{\pi})\bar{\pi}\sigma_{\epsilon(t+1)}\sigma_{\rho^N(t+1)} \\
&\quad - 2c_{\eta(t+1),\rho^N(t+1)}(1+r)r^F\bar{\pi}\sigma_{\eta(t+1)}\sigma_{\rho^N(t+1)} \\
&\quad + 2c_{\rho(t+1),\epsilon(t+1)}(1+r)b_\rho(t)\sigma_{\rho(t+1)}(a_\epsilon(t-1,t) - \bar{\pi})\sigma_{\epsilon(t+1)} \\
&\quad - 2c_{\rho(t+1),\eta(t+1)}r^F(1+r)b_\rho(t)\sigma_{\rho(t+1)}\sigma_{\eta(t+1)} \\
&\quad - 2c_{\epsilon(t+1),\eta(t+1)}r^F(a_\epsilon(t-1,t) - \bar{\pi})\sigma_{\epsilon(t+1)}\sigma_{\eta(t+1)}\Big) \\
&\quad + 2cov(\cdot, (\gamma F(t) + \eta(t+1))^-)
\end{aligned} \tag{17}$$

where the last component refers synthetically to the covariance of $(\gamma F(t) + \eta)^-$ with all other risk factors in the model (see Appendix A).

The unconstrained maximiser (dropping the time index for brevity) is given by (easy to solve because of a quadratic form):

$$\bar{\pi}^* = \arg\max_{\bar{\pi} \in \mathbf{R}}\{\mu_{R^c(t)} - \beta\Sigma^2_{R^c(t)}\} \tag{18}$$

The capital constraint based on the VaR formula remain complicated enough even in the "Gaussian world". It does not have a closed form solution as in the case of the liquidity constraint. However, we use the usual moment matching technique to yield a tractable formula for the constraint imposed on the lending choice. Notably, we match the moments of the conditional net income

$$NI_t := \mathbf{E}[K(t+1) - K(t)|\mathcal{F}_t] = \mathbf{E}[R(t+1)K(t)|\mathcal{F}_t]$$

with a normal distribution. The advantages of this particular choice of the distribution are twofold. First, since NI_t is a mixture of normal and log-normal distributions both tails stretch to infinity which of course is true also for the normally distributed random variable. Second, percentiles of the normal distribution are straightforward to compute. On the downside, the normal distribution does not allow controlling for the skewness of NI which may have important consequences for underestimation of the VaR numbers.

To approximate the VaR capital constraint we use the mean $\mu_{R^c(t)}$ and standard deviation of return $\sigma_{R^c(t)}$ defined in formulas 16 and 17 but with mean μ_X and risk σ_X parameters replaced by $\mu_{X,K}$ and $\sigma_{X,K}$ respectively. The approximation of the VaR constraint yields

$$\begin{aligned}
\mathrm{VaR}_{\alpha^K}\left(\frac{\mathrm{NI}_t - K(t)\mu_{R^c(t)}}{K(t)\sigma_{R^c(t)}}\right) &> \frac{-K(t) - K(t)\mu_{R^c(t)}}{K(t)\sigma_{R^c(t)}} \\
\Phi^2(\alpha^K) < \frac{(K(t)+K(t)\mu_K(t))^2}{K^2(t)\sigma^2_{R^c(t+1)}} &\quad \wedge \quad K(t)(1+\mu_{R^c(t+1)}) > 0 \\
-\Phi^2(\alpha^K)K^2(t)\sigma^2_R(t) &+ \left(K(t)+K(t)\mu_K(t)\right)^2 > 0
\end{aligned}$$

For a notational convenience let us decompose the mean and variance of R into terms at the optimal strategy and the residual terms.

$$\begin{aligned}
A_0^\mu &= (1+r)b_\rho(t)\mu_{\rho(t+1),K} - b_\rho(t) + a_\epsilon(t-1,t)(\mu_{\epsilon(t+1),K} - 1) \\
&\quad -r^F(F(t) + \gamma F(t) + \mu_{\eta(t+1),K}) + \frac{h}{(1-h)\sqrt{2\Pi}\sigma_{\eta(t+1),K}}(\gamma F(t) I_\eta^{(0)} + I_\eta^{(1)}) \\
A_\pi^\mu &= (1+r)\mu_{\rho^N,K} - 1 - (\mu_{\epsilon(t+1),K} - 1)
\end{aligned} \tag{19}$$

$$\begin{aligned}
A_0^\sigma = & \left((1+r)b_\rho(t)\sigma_{\rho(t+1),K}\right)^2 + a_\epsilon^2(t-1,t)\sigma^2_{\epsilon(t+1),K} + \left(r^F\sigma_{\eta(t+1),K}\right)^2 \\
+ & \frac{h^2}{(1-h)^2\sqrt{2\Pi}\sigma_{\eta,K}}\left((\gamma F(t))^2 I_\eta^{(0)} + 2\gamma F(t) I_\eta^{(1)} + I_\eta^{(2)}\right) \\
- & \left(\frac{h}{(1-h)\sqrt{2\Pi}\sigma_{\eta,K}}(\gamma F(t) I_\eta^{(0)} + I_\eta^{(1)})\right)^2 \\
+ & 2c_{\rho(t+1),\epsilon(t+1)}(1+r)b_\rho(t)\sigma_{\rho(t+1),K}a_\epsilon(t-1,t)\sigma_{\epsilon(t+1),K} \\
- & 2c_{\rho(t+1),\eta(t+1)}r^F(1+r)b_\rho(t)\sigma_{\rho(t+1),K}\sigma_{\eta(t+1),K} \\
& -2c_{\epsilon(t+1),\eta(t+1)}r^F a_\epsilon(t-1,t)\sigma_{\epsilon(t+1),K}\sigma_{\eta(t+1),K}) \\
- & \frac{(1+r)b_\rho(t)}{\sqrt{2\Pi}\sigma_{\eta,K}}\frac{h}{1-h}\gamma F\exp(c_2^2/2)I^{(0)}(-\gamma F, c_0 - \frac{\mu^2_{\eta,K}}{2\sigma^2_{\eta,K}}, c_1 + \frac{\mu_{\eta,K}}{\sigma^2_{\eta,K}}, -\frac{1}{2\sigma^2_{\eta,K}}) \\
- & \frac{(1+r)b_\rho(t)}{\sqrt{2\Pi}\sigma_{\eta,K}}\frac{h}{1-h}\exp(c_2^2/2)I^{(1)}(-\gamma F, c_0 - \frac{\mu^2_{\eta,K}}{2\sigma^2_{\eta,K}}, c_1 + \frac{\mu_{\eta,K}}{\sigma^2_{\eta,K}}, -\frac{1}{2\sigma^2_{\eta,K}}) \\
+ & \frac{(1+r)b_\rho(t)}{\sqrt{2\Pi}\sigma_{\eta,K}}\frac{h}{1-h}\mu_{\rho,K}(\gamma F(t) I_\eta^{(0)} + I_\eta^{(1)}) \\
+ & \frac{h a_\epsilon(t-1,t)}{\sqrt{2\Pi}\sigma_{\eta,K}(1-h)}\left[\bar{c}_0\gamma F I_\eta^{(0)} + \bar{c}_0 I_\eta^{(1)} + \bar{c}_1\gamma F I_\eta^{(1)} + \bar{c}_1 I_\eta^{(2)} - \mu_{\epsilon,K}\left(\gamma F(t) I_\eta^{(0)} + I_\eta^{(1)}\right)\right]
\end{aligned} \tag{20}$$

$$
\begin{aligned}
A_{\pi}^{\sigma} &= -2a_{\epsilon}(t-1,t)\sigma_{\epsilon(t+1),K}^{2} + 2c_{\rho(t+1),\rho^N(t+1)}(1+r)^2 b_{\rho}(t)\sigma_{\rho(t+1),K}\sigma_{\rho^N(t+1),K} \\
&+ \quad 2c_{\epsilon(t+1),\rho^N(t+1)}(1+r)a_{\epsilon}(t-1,t)\sigma_{\epsilon(t+1),K}\sigma_{\rho^N(t+1),K} \\
&- \quad 2c_{\eta(t+1),\rho^N(t+1)}(1+r)r^F\sigma_{\eta(t+1),K}\sigma_{\rho^N(t+1),K} \\
&- \quad 2c_{\rho(t+1),\epsilon(t+1)}(1+r)b_{\rho}(t)\sigma_{\rho(t+1),K}\sigma_{\epsilon(t+1),K} \\
&+ \quad 2c_{\epsilon(t+1),\eta(t+1)}r^F\sigma_{\epsilon(t+1),K}\sigma_{\eta(t+1),K}) \\
&- \quad \frac{1+r}{\sqrt{2\Pi}\sigma_{\eta,K}}\frac{h}{1-h}\gamma F\exp(c_2^2/2)I^{(0)}(-\gamma F, c_0 - \frac{\mu_{\eta,K}^2}{2\sigma_{\eta,K}^2}, c_1 + \frac{\mu_{\eta,K}}{\sigma_{\eta,K}^2}, -\frac{1}{2\sigma_{\eta,K}^2}) \\
&- \quad \frac{1+r}{\sqrt{2\Pi}\sigma_{\eta,K}}\frac{h}{1-h}\exp(c_2^2/2)I^{(1)}(-\gamma F, c_0 - \frac{\mu_{\eta,K}^2}{2\sigma_{\eta,K}^2}, c_1 + \frac{\mu_{\eta,K}}{\sigma_{\eta,K}^2}, -\frac{1}{2\sigma_{\eta,K}^2}) \\
&+ \quad \frac{1+r}{\sqrt{2\Pi}\sigma_{\eta,K}}\frac{h}{1-h}\mu_{\rho^N,K}(\gamma F(t)I_{\eta}^{(0)} + I_{\eta}^{(1)}) \\
&- \quad \frac{h}{\sqrt{2\Pi}\sigma_{\eta,K}(1-h)}\left[\bar{c}_0\gamma F I_{\eta}^{(0)} + \bar{c}_0 I_{\eta}^{(1)} + \bar{c}_1\gamma F I_{\eta}^{(1)} + \bar{c}_1 I_{\eta}^{(2)} - \mu_{\epsilon,K}\left(\gamma F(t)I_{\eta}^{(0)} + I_{\eta}^{(1)}\right)\right] \\
A_{\pi\pi}^{\sigma} &= \Big((1+r)\sigma_{\rho^N(t+1),K}\Big)^2 + \sigma_{\epsilon(t+1),K}^2 - 2c_{\epsilon(t+1),\rho^N(t+1)}(1+r)\sigma_{\epsilon(t+1),K}\sigma_{\rho^N(t+1),K}
\end{aligned}
$$

Consequently, the VaR constraint can be reformulated to

$$-\Phi^2(\alpha^K)(A_0^{\sigma} + A_{\pi}^{\sigma}\bar{\pi} + A_{\pi\pi}^{\sigma}\bar{\pi}^2) + (K(t) + A_0^{\sigma} + A_{\pi}^{\mu}\bar{\pi})^2 > 0$$

and by grouping of terms to

$$\left[-\Phi^2(\alpha^K)A_{\pi\pi}^{\sigma} + (A_{\pi}^{\mu})^2\right]\bar{\pi}^2 + \left[-\Phi^2(\alpha^K)A_{\pi}^{\sigma} + 2A_{\pi}^{\mu}(K(t) + A_0^{\mu})\right]\bar{\pi} + \left[-\Phi^2(\alpha^K)A_0^{\sigma} + (K(t) + A_0^{\mu})^2\right] > 0$$

For many reasonable parameterisations of the model, the coefficient at $\bar{\pi}^2$ is negative. It means that the admissible investment would be bounded from above. Notably, it is theoretically possible that the strategy is bounded from below by a positive number meaning that the set of admissible strategies can be disjoint; the bank is not allowed to invest too much in the risky loans but may be forced to allocate a minimum part of the reinvestment portfolio for profitability reasons (i.e. bank could be solvent only if earning a sufficient income from profitable loans).

4 2-period setup

The general multi-period model can be costly in terms of computational time. We focus on the applications of a 2-period model. It only requires approximate (Monte-Carlo) methods to be implemented in the first period whereas at the second (and final) period the solution is explicit. Notably, it preserves all the important features of T-period model; the inter-temporal effects resulting in a trade-off between investing more in profitable loans now and facing risk of illiquidity or generating less income but increasing survival probability.

The optimisation problem is solved numerically by means of the dynamic programming. It is a convenient way to derive the optimal portfolios in a backward manner. The value function of the 2-period model at time $t = 2$ is equal to 0, i.e.

$$V_2 \equiv 0 \tag{21}$$

since this is the end of investment horizon and, by definition, no income is generated afterwards. At $t = 1$ the problem is solved analytically and the optimal portfolio choice at $t = 0$ is calculated as value π_0^* that maximises the sum of average of the value function over the sampled portfolios at $t = 1$ and the utility of loan investment π_0^*.

At $t = 0$ we define random functions (randomness implied by $\epsilon(1)$, $\rho(1)$, $\rho^N(1)$ and $\eta(1)$):

$$
\begin{aligned}
l(l_0, \bar{\pi}) &= (1-m)l_0\rho(1) + \bar{\pi}\rho^N(1) \\
s(s_0, l_0, f_0, f_0, \bar{\pi}) &= ml_0 + s_0 + r((1-m)l_0\rho(1) + \bar{\pi}) + f_0 - f_0 - \bar{\pi})\epsilon(1) \\
f(f_0) &= f_0 + \gamma f_0 + \eta(1) \\
k(s_0, l_0, f_0, f_0, \bar{\pi}) &= (l_0 + s_0 - f_0) + rl(l_0, \bar{\pi}) - r^F f(f_0) + \Delta L(t) + \Delta S(t) - \frac{h}{1-h}(f(f_0) - f_0)^-
\end{aligned}
$$

The value function at $t = 0$ satisfies the following dynamic programming formula

$$
\begin{aligned}
V_0(l_0, s_0, f_0) = \max_{\bar{\pi}\in\mathcal{A}(0)} \Big\{ &\mathbf{E}[R(1) - \beta Var(R(1))] + \\
- \quad & \delta\phi(l(l_0, \bar{\pi}) + s(s_0, l_0, f_0, f_0, \bar{\pi}))\mathbb{I}_{\{\tau=1\}} + \delta V_1(l(l_0, \bar{\pi}), s(s_0, l_0, f_0, f_0, \bar{\pi}), f_0, f(f_0)) \Big\}
\end{aligned}
$$

and applying 16 and 17 it can be transformed to

$$
\begin{aligned}
V_0(l_0, s_0, f_0) = \max_{\bar{\pi}\in\mathcal{A}(0)} \Big\{ &\mu(l_0, s_0, f_0, f_0, \bar{\pi}) - \beta\Sigma^s(l_0, s_0, f_0, f_0, \bar{\pi}) \\
&-\delta\phi\mathbf{E}\Big[(l(l_0, \bar{\pi}) + s(s_0, l_0, f_0, f_0, \bar{\pi}))\mathbb{I}_{\{k(s_0,l_0,f_0,f_0,\bar{\pi})<0 \,\vee\, (1-h)s(s_0,l_0,f_0,f_0,\bar{\pi})<f(f_0)-f_0\}}\Big] \\
&+\delta\mathbf{E}V_1(l(l_0, \bar{\pi}), s(s_0, l_0, f_0, f_0, \bar{\pi}), f_0, f(f_0)) \Big\} \qquad (22)
\end{aligned}
$$

The function V_0 is well-defined and well-behaved since V_1 is continuous and it can be shown that $g : (0, +\infty]^4 \to \mathbf{R}_+$ such that $g(l_0, s_0, f_0, \bar{\pi}) : = \mathbf{E}\mathbb{I}_{\{\tau=1\}}$ is also continuous.[12] Hence, max is attained on a compact set $\mathcal{A}(0)$.

The form of equation 22 is not very specific to the 2-period setup of the model. It can be straightforwardly generalised to a multi-period case, whereby the value function at any $t < T$ is identical to 22 after replacing 0 and 1 time indices with t and $t + 1$ respectively.

5 Simulations

We choose a specific parametrisation of the model summarised in the table 1. The two important comments to the table are related to the selection of the LaR and VaR horizon (Δ_l and Δ_K respectively) and to the credit risk parameters. First, we assume that $\Delta_l = 0.08(3)$, corresponding to 30 days in a 360-day year investment horizon, and $\Delta_K = 0.25$ which corresponds to 90 days for the solvency constraint. Second, the parameters for the average and standard deviation of defaults on the new loans (ρ^N) having by assumption the log-normal distribution are presented in terms of the means and standard deviation of normal distribution generating that log-normal distribution. After transformation (given by formulas 13 and 14), the mean and standard deviation of ρ^N are 5.3% and 8.0% respectively. Taking into account the interest rate on loans, the Sharpe ratios of loans and securities can be calculated to compare the relative risk-adjusted return of both investment opportunities available to the bank (the ration for loans amounts to 0.59 against 0.50 for securities so the risk-return profile of the two categories is relatively balanced).

Case 1: Optimal asset structure for different levels of funding. The optimal lending volume is calculated for funding volumes ranging between 93.0 and 93.8, all other parameters unchanged except implicitly changing capital level from 7.0 to 6.8. This can be interpreted as an increase in leverage ratio. Figure 1 presents a distribution of CAR projected one period ahead using the optimal lending strategy for different levels of funding. The upper-left pane of the graph plot the admissible region of investment in new loans against different levels of funding. The region is delimited by the intersection of CAR-based, VaR-based and liquidity constraints. A straightforward observation is

[12] In fact, a sufficient condition for g to be continuous is that ϵ, η, ρ and ρ^N have continuous cumulative distribution functions.

Table 1: Example of parametrisation of the model

Parameter	Value	Description	Parameter	Value	Description
m	0.10	maturity	μ_ϵ	1.03	mean return on S
r	0.07	interest rate of L	σ_ϵ	0.06	std of return on S
r^F	0.03	cost of funding	μ_Z	-3.55	mean default factor (outstanding L)
h	0.05	haircut	σ_Z	1.10	std default factor (outstanding L)
γ	0.02	trend of funding	μ_{Z^N}	-3.20	mean default factor (new L)
κ	0.08	capital constraint	σ_{Z^N}	1.25	std default factor (new L)
α^F, α^K	0.01	liq., solv. constraints	μ_η	0.00	mean of funding risk
β	5.00	risk aversion	σ_η	10.00	std of funding risk
w^L	0.85	risk weight (L)	$c_{\epsilon,Z}$	0.00	corr(securities,default of outstanding L)
w^S	0.20	risk weight (S)	c_{ϵ,Z^N}	0.00	corr(securities,default of new L)
δ	0.70	discount factor	$c_{\epsilon,\eta}$	0.00	corr(securities,funding)
L	70.00	loans	c_{Z,Z^N}	0.00	corr(outstanding L, new L
S	30.00	securities	$c_{Z,\eta}$	0.00	corr(default of outstanding L, funding)
F	93.00	funding	$c_{Z^N,\eta}$	0.00	corr(default of new L, funding)
ϕ	0.00	penalty rate		.	

Source: own calculations

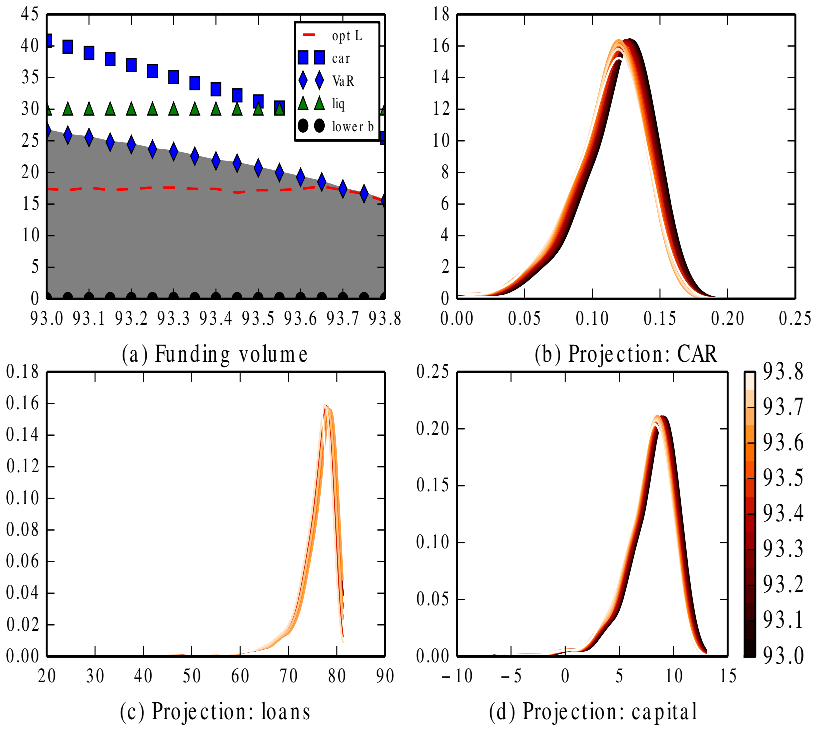
that the lending remains stable until the VaR-based capital constraint becomes a binding constraint. Clearly, the increasing leverage implies that a bank can invest less in VaR consuming loans. The whole distribution of the capital ratios shifts to the left (upper-right pane). Notably, due to the fire-sales the distribution is also skewed to the left.

Case 2: Optimal asset structure for different levels of funding risk. An analogous simulation to the Case 1 was conducted assuming that the volatility parameter of the funding (σ^F) was varied in the range of 10 to 20. It impacts the maximum allowed reinvestment into loan portfolio by the liquidity constrain, whereas the CAR capital constraint remains flat (insensitive to the risk parameters on the funding side). For the low volatility of the funding, as the volatility starts to increase the optimal lending portfolio slides down slowly and stays below the potential level (implied by liquidity and CAR constraints). At a certain level of volatility (13 units in the example), liquidity constraint becomes binding and afterwards the optimal lending follows the path implied by the liquidity condition. Interpreting the results form the angle of the credit provision to the economy, if the volatility of funding is relatively low then the optimal lending is immune to the changes of the volatility of the funding sources until the funding risk becomes a binding constraint from the solvency perspective.

Case 3: Riskiness of securities portfolio. Portfolio allocation may vary substantially if the relative riskiness of loan and securities (liquidity) portfolio changes (between 6% and 11.5%). We verified the theoretically optimal structure of banks assets under different volatility of the securities (see 3). Apparently, rising volatility leads to a shift of allocation towards the loan portfolio. The classical Sharp ratio matters for the choice and the liquidity constraint seems not to be binding in this case. However, the VaR-based capital constraint prevails severely reducing the lending potential of the bank as the volatility grows. For σ^S below 10.0% the constraints are not binding and allocation to the loan portfolio increases but above the 10.0% threshold the VaR constraint becomes binding and the optimal investment in the new loans slides down slightly along the VaR solvency constraint. Notably, the decrease in lending is more sensitive to changes in the volatility of securities if the correlation between returns from securities and loan losses is positive. The graph 4 shows that the correlation amplifies the combined risk in the banks's assets and high volatility of S translates into lower potential for lending. Moreover, the higher market and credit risk find a compensation in the rising lower bound for investment in the new loans which offer higher expected interest payments. Consequently, the CAR oscillates more as well (the density of the distribution flattens on the upper-right pane).

It is interesting to see the impact of the correlation between securities and funding. A highly positive correlation between returns from securities and the funding volumes – a more unfavorable configuration from the liquidity perspective – does not change much the picture qualitatively (5). Though, the capital constraint is less restrictive and allows loan portfolios to grow more in high

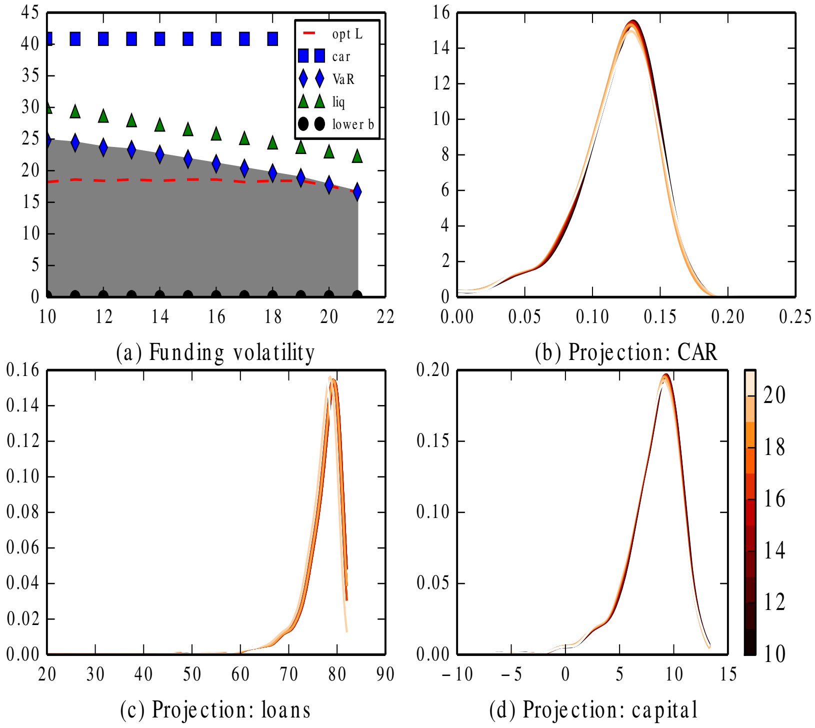
Figure 1: Sensitivity of the optimal balance sheet structure and its capitalisation to leverage ratio



Note: opt L: optimal lending volume; car: maximum lending volume allowed by the CAR ratio with a given level of capital; liq: maximum lending allowed by the liquidity constraint; reinv: reinvested part of the balance sheet, i.e. $(1-m)L+S$. The shaded area shows the region of admissible investment in the new loan portfolio volumes. The density plots present kernel density estimates of projected at the end of the first period, optimal levels of (b) CAR, (c) loan volumes and (d) capital

Source: own calculations, graphs generated using *Matplotlib* in Python

Figure 2: Capital Adequacy Ratio distribution for various levels of funding risk

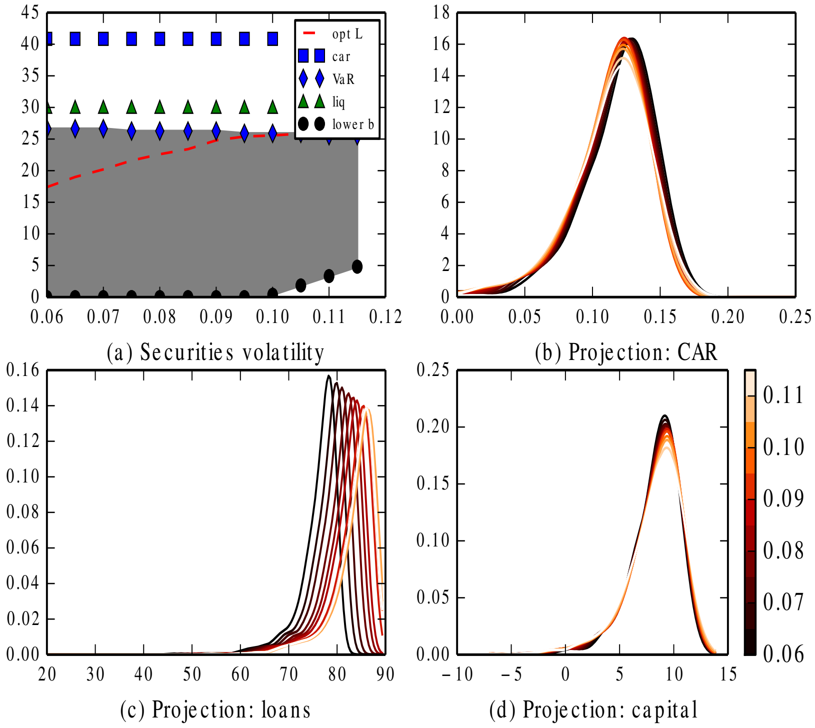


(a) Funding volatility (b) Projection: CAR

(c) Projection: loans (d) Projection: capital

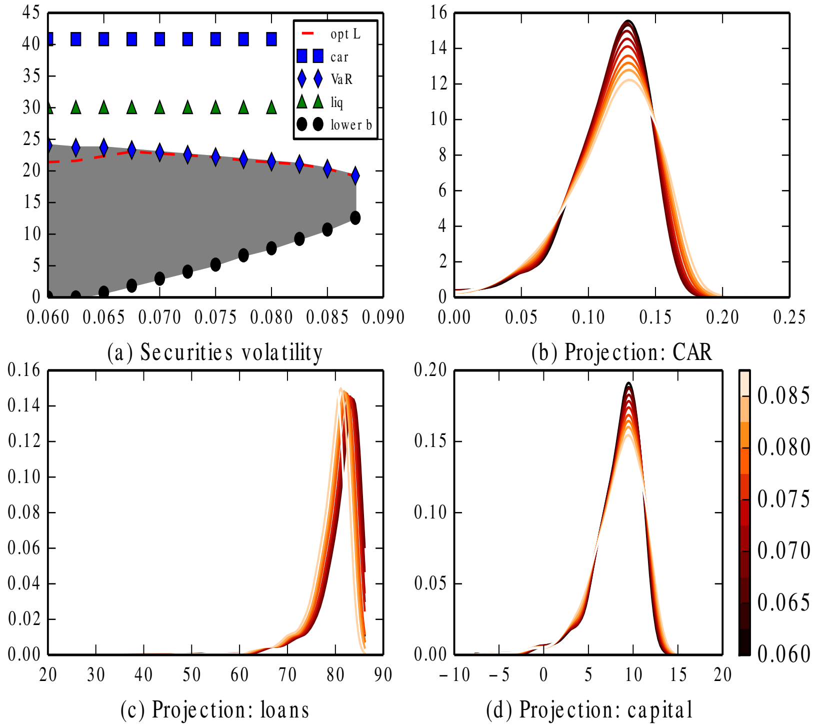
Source: own calculations, graphs generated using *Matplotlib* in Python

Figure 3: Capital Adequacy Ratio distribution for various levels of volatility of securities



Source: own calculations, graphs generated using *Matplotlib* in Python

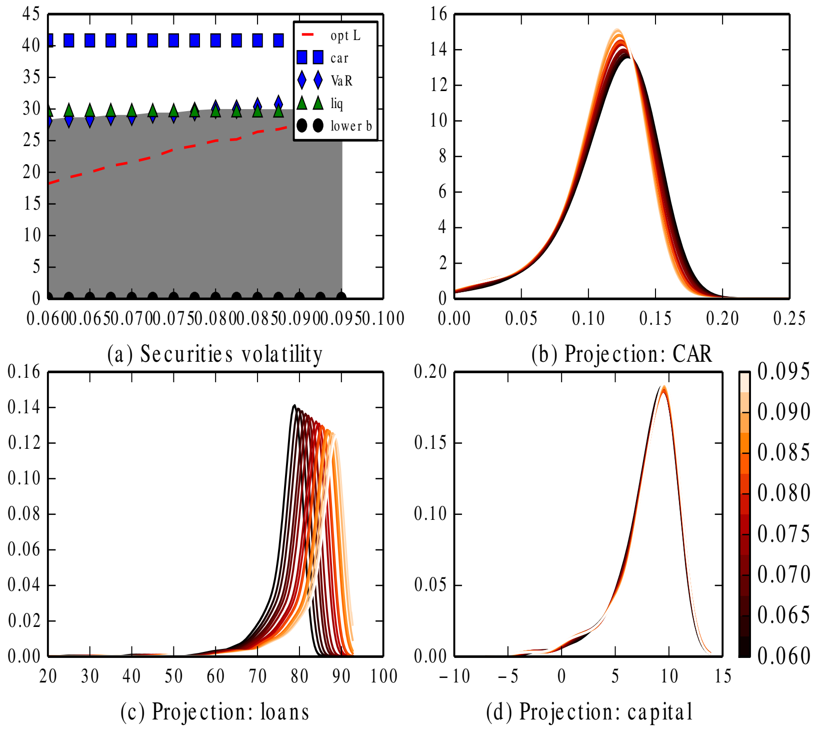
Figure 4: Capital Adequacy Ratio distribution for various levels of volatility of securities and positive correlation between loan losses and returns from securities



Note: Positive correlation was set by $c_{\epsilon,\rho^N} = -0.5$
Source: own calculations, graphs generated using *Matplotlib* in Python

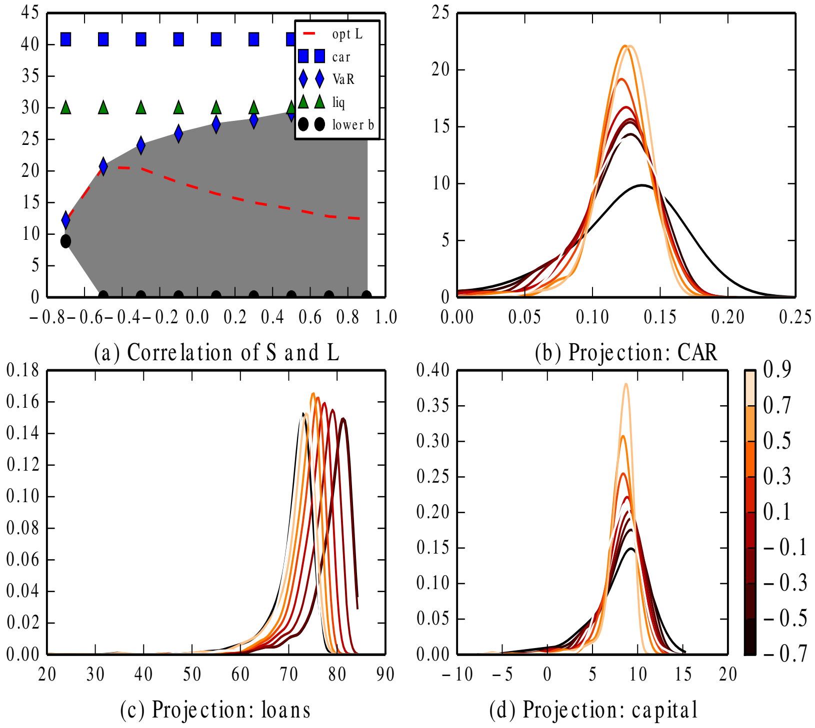
correlation regime.
Case 4: Correlation of the credit and market risk. The correlation between market risk (valuation of securities) and credit risk have a substantial impact on the optimal allocation of banks' assets. Negative correlation narrows the investment opportunities for the banks in the model. Negative correlation means that the devaluation of securities' portfolios is likely to coincide with the increases in loan losses. That translates to a binding VaR solvency constraint for very negative correlation. Nevertheless, the rising negative correlation implies more investment in the loan portfolios. As the dependence between market and credit losses becomes weaker the optimum indicate less lending and more investment in the securities. Therefore, the relationship between propensity to lending and correlation between asset classes is very nonlinear. Correlation is an extremely important factor of the decision process.
Case 5: Return on securities. The return on securities, and in fact the relationship between returns on securities and loans, is in our model a significant driver of bank's optimal choice of the asset structure. As the returns on securities rise, the optimal allocation in the loan portfolio starts to decrease when the return on securities reaches a certain level and steadily diminishes until the full allocation in the securities portfolio is achieved. At the same time, capital level decline gradually and become more oscillatory (bottom-right pane). However, capital adequacy increases since risk-weighted assets drop materially (upper-right pane).
Case 6: Fire-sales. The outflow of deposits triggers a liquidation of the securities in the securities portfolio S. The model allows for studying of the impact of the devaluation parameter (the haircut h) on the optimal lending, on the bank's profitability and on the financial soundness of a bank

Figure 5: Capital Adequacy Ratio distribution for various levels of volatility of securities, with correlation of ϵ and η equal to 0.8



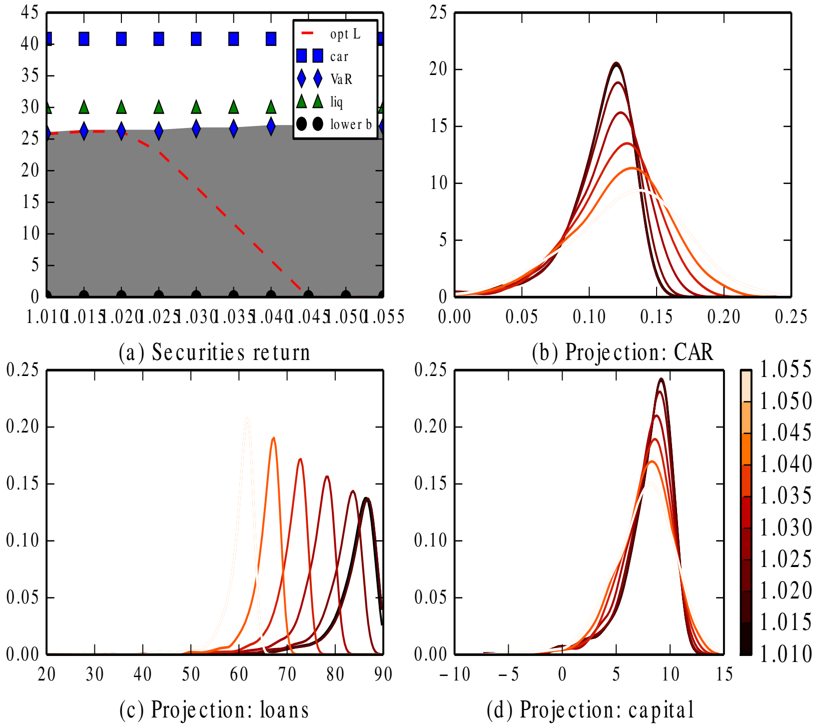
Source: own calculations, graphs generated using *Matplotlib* in Python

Figure 6: Capital Adequacy Ratio distribution for various levels of correlation between credit risk and market risk



Source: own calculations, graphs generated using *Matplotlib* in Python

Figure 7: Capital Adequacy Ratio distribution for various levels of return on securities



Source: own calculations, graphs generated using *Matplotlib* in Python

(liquidity and solvency). As an illustration, the optimal program of the bank was solved for different parameters of the haircut – from the baseline 1% to 10%. The haircut of 10% reflects a sizeable impact of the sell-off of the securities that would be required to meet the obligations related to an outflow of deposits. The lending profile is non-linear with respect to the haircut which is primarily related to the solvency constraint which becomes binding for the sufficiently large values of the haircut, i.e. above 8%. Below that threshold the theoretically optimal volume of lending slightly increases with the haircut. It can be interpreted as seeking for relatively more profitable loans. Notably, the liquidity constraint is only to a very little extent affected by the increasing haircut (and is never binding in the specific example). The haircut does not significantly affect the capitalisation of the banks (a stable distribution of CAR on Figure 8, bottom-right pane) even though the impact on the valuation of the securities is visible (top-right pane of Figure 9). Conversely, the level of haircut (i.e. liquidity of the counterbalancing securities portfolio) has a significant influence on the risk of bank's liquidity and solvency default, illustrated on Figure 10. Intuitively, the solvency risk rises as the cost of liquidation increases (implying higher erosion of the capital base). Analogously, the bank is becoming more likely to default on the obligations towards the depositors until the haircut renders the solvency a binding constraint. From that point the optimal lending volume declines which is equivalent to a higher build-up of the liquid securities playing a role of the liquidity buffer.

The liquidity brings a non-trivial dynamics to the model and provides an insight into the complicated behaviors of banks. The interplay of the solvency and liquidity has some interesting policy implications. Let us suppose that a regulator wants to improve market liquidity and reduces the general liquidation haircut of securities from 10% to 8%. By assumption, these securities comprise the liquidity portfolio of the bank characterised by the parameters in Table 1. Of course, it is a very stylised and specific configuration but the set of parameters is consistent and realistic. Consequently, the bank is induced to increase lending at the optimum and its solvency improves in terms of the probability that its capital ratio stays above the regulatory minimum. However, its liquidity buffer declines and the bank become more prone to the liquidity risk (measured as the likelihood that the outflow of the funding sources exceeds the liquidity buffer). The example shows that any policy targeting either solvency or liquidity of the bank has to be calibrated and adopted with caution to minimise the adverse externality it can cause – a regulatory action to contain one type of risk can be harmful for a different type.

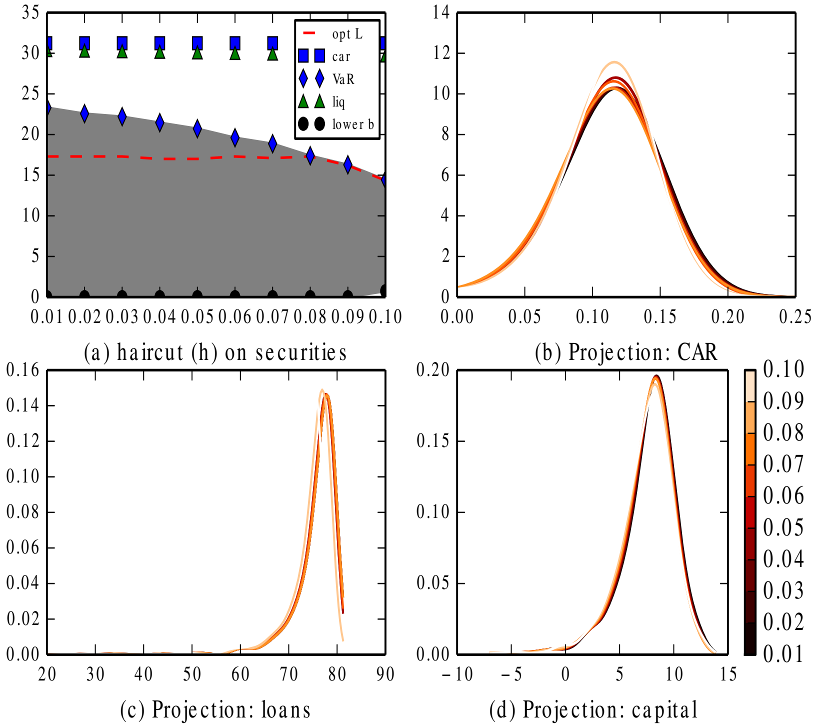
Case 7: Bankruptcy costs. We have set the bankruptcy costs ϕ to 0 to distil the risk-adjusted return effects on the optimal portfolio reallocation from a more arbitrary estimate of the aversion against insolvency and illiquidity risk. Intuitively, the Figure 11 shows that the optimal lending declines as the fraction ϕ, representing the bankruptcy cost as a fraction of the total assets, rises. Though, the sensitivity is rather moderate. Notably, the average capital ratio increases with the the penalty rate but the dispersion of CAR as well meaning that inadequate values are more likely. From the policy perspective this result indicates that the provision of credit is, to some extent sensitive to the bank resolution scheme and its efficiency. It may influence the behaviour of banks and their shareholders depending on the expected cost of bankruptcy.

6 Conclusions

The paper presents a stylised model of the optimal structure of a simplified balance sheet of a bank. It models a balance sheet composed of two asset classes finance by homogenous risky funding. The decision of a bank is to find the level of lending to strike a balance between return form investment into illiquid loans and liquid securities offering counterbalancing capacity against funding risk (i.e. fluctuation of available funding volumes). The implementation of the model is comprehensive in how it treats the return-vs-risk trade-off and regulatory aspects of the banking system. Moreover, it is easy to solve by the Monte Carlo simulations and the complex, VaR-based risk constraints are reduced to the closed-form, analytical formulas.

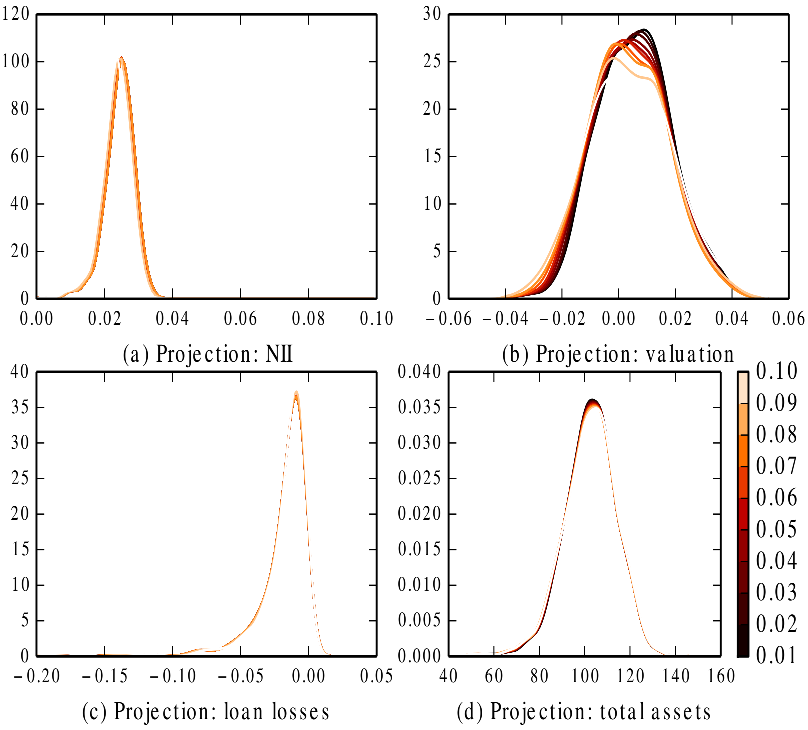
The setup of the model straightforwardly suggests some interesting applications in the macro-

Figure 8: Capital Adequacy Ratio distribution for various levels of the fire-sales haircut on securities



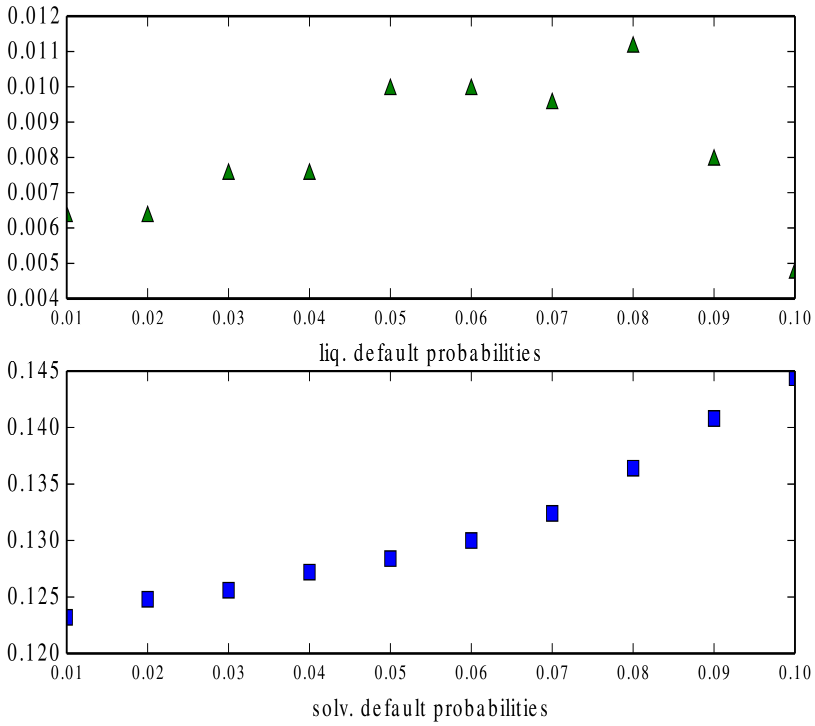
Source: own calculations, graphs generated using *Matplotlib* in Python

Figure 9: Decomposition of capital and P&L drivers for various levels of the fire-sales haircut on securities



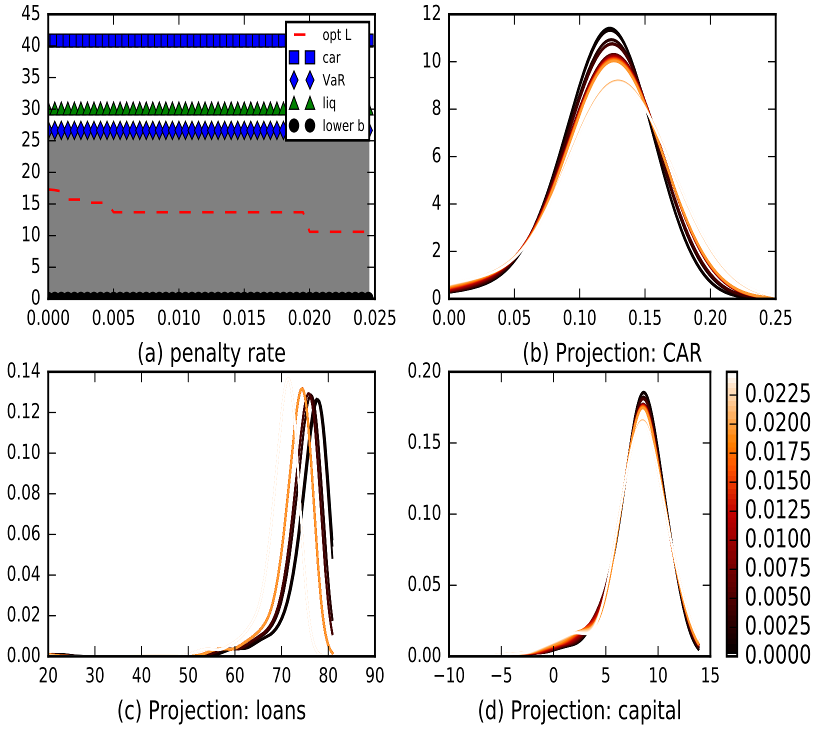
Source: own calculations, graphs generated using *Matplotlib* in Python

Figure 10: Realised default frequency for various levels of the fire-sales haircut on securities



Source: own calculations, graphs generated using *Matplotlib* in Python

Figure 11: Capital Adequacy Ratio distribution for various levels of bankruptcy costs



Note: A rather step-wise monotonicity of the optimal lending as a function of ϕ is a result of the numerical approximation of the optimal solution and not a feature of the optimal solution per se.
Source: own calculations, graphs generated using *Matplotlib* in Python

prudential policy assessment; it can be used to analyse potential strategic responses of banks to macro-prudential capital buffers, portfolio specific risk weights or liquidity requirements.

The current 2-asset model in 2 periods can be extended easily to n-asset model, however with a loss of closed form portfolio strategy in the second period. This would massively complicate the calculations. Consequently, some pure Monte-Carlo techniques would need to be applied and a substantial increase in the computation time (although not unmanageable) should be expected.

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A Integrals

Let us solve the expression $I^{(0)}(a, B_0, B_1, B_2)$. Applying the fact that for $f \in \mathcal{C}^1$, $f(a_1) - f(a_2) = \int_{a_1}^{a_2} f'(x)\mathrm{d}x$ leads to

$$I^{(0)}(a, B_0, B_1, B_2) = \frac{\sqrt{2\Pi}}{\sqrt{-2B_2}} \exp\left(\frac{B_1^2}{-4B_2} + B_0\right) \Phi\left(a\sqrt{-2B_2} - \frac{B_1}{\sqrt{-2B_2}}\right) \tag{23}$$

Consequently,

$$I^{(1)}(a, B_0, B_1, B_2) = \left([\exp(B_2x^2 + B_1x + B_0)]\Big|_{-\infty}^{a} - B_1 \int_{-\infty}^{a} \exp(B_2x^2 + B_1x + B_0) \right) / (2B_2)$$

$$= \left(\exp(B_2a^2 + B_1a + B_0) - B_1 \frac{\sqrt{2\Pi}}{\sqrt{-2B_2}} \exp\left(\frac{B_1^2}{-4B_2} + B_0\right) \Phi\left(a\sqrt{-2B_2} - \frac{B_1}{\sqrt{-2B_2}}\right) \right) / (2B_2)$$

and again integrating by parts

$$\begin{aligned} I^{(2)}(a, B_0, B_1, B_2) &= \int_{-\infty}^{a} xx \exp(B_2x^2 + B_1x + B_0)\mathrm{d}x \\ &= \frac{1}{2B_2}\left(xe^{B_2x^2+B_1x+B_0}\Big|_{-\infty}^{a} - \int_{-\infty}^{a} \exp(B_2x^2 + B_1x + B_0)\mathrm{d}x \right) \\ &\quad - \frac{B_1}{2B_2} \int_{-\infty}^{a} x \exp(B_2x^2 + B_1x + B_0)\mathrm{d}x \\ &= \frac{a}{2B_2} e^{B_2a^2+B_1a+B_0} - (2B_2)^{-1} I^{(0)}(a, B_0, B_1, B_2) - \frac{B_1}{2B_2} I^{(1)}(a, B_0, B_1, B_2) \end{aligned}$$

Let us consider the covariance of terms involving $-(\gamma F(t) + \eta(t))^-$ and Z_N (or analogously Z). Since Z_N and $\eta(t)$ are jointly Gaussian, Z_N can be represented as an affine combination of $\eta(t)$ and some independent, normally distributed variable u, $\mathbf{E}u = 0$ and $\sigma_u = 1$, as

$$Z_N = c_0 + c_1\eta(t) + c_2u$$

Consequently,

$$\sigma_Z^2 = c_1^2\sigma_\eta^2 + c_2^2$$

$$\begin{aligned} \mu_{Z_N} &= c_0 + c_1\mu_\eta \\ c_{Z_N,\eta(t)} &= \frac{cov(Z_N, \eta(t))}{\sigma_{Z_N}\sigma_\eta} = \frac{c_1\sigma_\eta}{\sigma_{Z_N}} \end{aligned}$$

Solving:

$$c_0 = \mu_{Z_N} - c_{Z_N,\eta}\frac{\sigma_{Z_N}}{\sigma_\eta}\mu_\eta, \quad c_1 = \frac{\sigma_{Z_N}}{\sigma_\eta}c_{Z_N,\eta}, \quad c_2 = \sigma_{Z_N}\sqrt{1 - c_{Z_N,\eta}^2}$$

Then:

$$
\begin{aligned}
&cov(-(1+r)e^{Z_N}, -\frac{h}{1-h}(\gamma F+\eta)^-) = (1+r)\frac{h}{1-h}cov(e^{Z_N}, (\gamma F+\eta)^-)\\
&= (1+r)\frac{h}{1-h}\mathbf{E}\int_{-\infty}^{-\gamma F} -\frac{1}{\sqrt{2\Pi}\sigma_\eta}(\gamma F+y)\exp(c_0+c_1y+c_2u)\exp\left(-\frac{(y-\mu_\eta)^2}{2\sigma_\eta^2}\right)\mathrm{d}y\\
&\quad -(1+r)\frac{h}{1-h}\mathbf{E}e^{Z_N}\mathbf{E}(\gamma F+\eta)^-\\
&= -\frac{1+r}{\sqrt{2\Pi}\sigma_\eta}\frac{h}{1-h}\gamma F\exp(c_2^2/2)I^{(0)}(-\gamma F, c_0-\frac{\mu_\eta^2}{2\sigma_\eta^2}, c_1+\frac{\mu_\eta}{\sigma_\eta^2}, -\frac{1}{2\sigma_\eta^2})\\
&\quad -\frac{1+r}{\sqrt{2\Pi}\sigma_\eta}\frac{h}{1-h}\exp(c_2^2/2)I^{(1)}(-\gamma F, c_0-\frac{\mu_\eta^2}{2\sigma_\eta^2}, c_1+\frac{\mu_\eta}{\sigma_\eta^2}, -\frac{1}{2\sigma_\eta^2})\\
&\quad +\frac{1+r}{\sqrt{2\Pi}\sigma_\eta}\frac{h}{1-h}\exp(\mu_{Z_N}+\frac{\sigma_{Z_N}^2}{2})(\gamma F(t)I_\eta^{(0)}+I_\eta^{(1)})
\end{aligned}
$$

where $I_\eta^{(n)}$ is defined in the context of equation 16. The covariance of $-(\gamma F(t)+\eta(t))^-$ and Z has almost exactly the same representation.

The final missing component of the covariance matrix is

$$cov((a_\epsilon(t-1,t)-\bar{\pi})\epsilon(t+1), -\frac{h}{1-h}(\gamma F(t)+\eta(t))^-)$$

we define

$$\bar{c}_0 = \mu_\epsilon - c_{\epsilon,\eta}\frac{\sigma_\epsilon}{\sigma_\eta}\mu_\eta, \quad \bar{c}_1 = \frac{\sigma_\epsilon}{\sigma_\eta}c_{\epsilon,\eta}, \quad \bar{c}_2 = \sigma_\epsilon\sqrt{1-c_{\epsilon,\eta}^2}$$

$$
\begin{aligned}
&cov((a_\epsilon(t-1,t)-\bar{\pi})\epsilon(t+1), -(\gamma F(t)+\eta(t))^-)\\
&= \frac{h(a_\epsilon(t-1,t)-\bar{\pi})}{1-h}\mathbf{E}\left(\frac{1}{\sqrt{2\Pi}\sigma_\eta}\int_{-\infty}^{-\gamma F}(\bar{c}_0+\bar{c}_1z+\bar{c}_2u)(\gamma F+z)\exp\left(-\frac{(z-\mu_\eta)^2}{2\sigma_\eta^2}\right)\mathrm{d}z\right)\\
&\quad +\frac{h(a_\epsilon(t-1,t)-\bar{\pi})}{1-h}\mathbf{E}\epsilon(t+1)\mathbf{E}(\gamma F(t)+\eta(t))^-)\\
&= \frac{h(a_\epsilon(t-1,t)-\bar{\pi})}{\sqrt{2\Pi}\sigma_\eta(1-h)}\left[\bar{c}_0\gamma F I_\eta^{(0)}+\bar{c}_0 I_\eta^{(1)}+\bar{c}_1\gamma F I_\eta^{(1)}+\bar{c}_1 I_\eta^{(2)}-\mu_\epsilon\left(\gamma F(t)I_\eta^{(0)}+I_\eta^{(1)}\right)\right]
\end{aligned}
$$

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